

# IOWA STATE UNIVERSITY

## Digital Repository

---

Retrospective Theses and Dissertations

Iowa State University Capstones, Theses and  
Dissertations

---

1990

## The asymptotic behavior of solutions of some nonlinear initial-boundary value problems of parabolic type

Keng Deng

*Iowa State University*

Follow this and additional works at: <https://lib.dr.iastate.edu/rtd>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Deng, Keng, "The asymptotic behavior of solutions of some nonlinear initial-boundary value problems of parabolic type" (1990). *Retrospective Theses and Dissertations*. 9436.  
<https://lib.dr.iastate.edu/rtd/9436>

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact [digirep@iastate.edu](mailto:digirep@iastate.edu).

3

91

01347

U·M·I

MICROFILMED 1990

## **INFORMATION TO USERS**

**The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.**

**The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.**

**In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.**

**Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.**

**Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.**

# **U·M·I**

University Microfilms International  
A Bell & Howell Information Company  
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA  
313 761-4700 800 521-0600



**Order Number 9101347**

**The asymptotic behavior of solutions of some nonlinear  
initial-boundary value problems of parabolic type**

**Deng, Keng, Ph.D.**

**Iowa State University, 1990**

**U·M·I**

**300 N. Zeeb Rd.  
Ann Arbor, MI 48106**



**The asymptotic behavior of solutions of some nonlinear  
initial-boundary value problems of parabolic type**

by

**Keng Deng**

**A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of the  
Requirements for the Degree of  
DOCTOR OF PHILOSOPHY**

**Department: Mathematics**

**Major: Applied Mathematics**

**Approved:**

**Members of the Committee:**

Signature was redacted for privacy.

**In Charge of Major Work**

Signature was redacted for privacy.

**For the Major Department**

Signature was redacted for privacy.

Signature was redacted for privacy.

**For the Graduate College**

**Iowa State University  
Ames, Iowa**

**1990**

## TABLE OF CONTENTS

<b>GENERAL INTRODUCTION</b>	<b>1</b>
<b>Explanation of Dissertation Format</b>	<b>2</b>
<b>PART I. ON THE BLOW UP OF <math>u_t</math> AT QUENCHING</b>	<b>3</b>
<b>Abstract</b>	<b>3</b>
1. Introduction	3
2. The Location of the Quenching Points	6
3. The Blow Up of $u_t$ at Quenching	9
4. The Blow Up of $u_t$ for the Robin Condition	11
5. References	14
<b>PART II. QUENCHING FOR SOLUTIONS OF A PLASMA TYPE EQUATION</b>	<b>15</b>
<b>Abstract</b>	<b>15</b>
1. Introduction	15
2. Comparison and Local Existence	16
3. Stationary Solutions	18
4. Quenching and Global Existence	30
5. References	36
<b>PART III. BEHAVIOR OF SOLUTIONS OF BURGERS' EQUATION WITH NONLOCAL BOUNDARY CONDITIONS</b>	<b>37</b>
<b>Abstract</b>	<b>37</b>
1. Introduction	37
2. Stationary Solutions and their Basic Properties	39
3. Comparison Theorem, Local Existence, and Blow Up of Solutions	60
4. Stability and Instability Analysis	74
5. References	89



<b>GENERAL SUMMARY</b>	<b>91</b>
<b>LITERATURE CITED</b>	<b>92</b>
<b>ACKNOWLEDGEMENTS</b>	<b>93</b>

## GENERAL INTRODUCTION

Throughout the development of calculus, various physical phenomena have been modeled into partial differential equations. One well-known equation is the heat equation  $u_t = \Delta u$ , which describes heat conduction under Fourier's Law. Of more interest are nonlinear problems, many of which arise in such fields as electronics, hydrodynamics, chemical kinetics, and biophysics ([1],[2],[4],[7]).

When a mathematician deals with a partial differential equation subject to either a purely initial condition or to initial-boundary conditions, he is usually first confronted with the following questions: Does there exist a solution? If so, is the solution unique and does it depend "continuously" on initial data? If answers to all these questions are affirmative, then the problem is said to be well-posed in the sense of Hadamard. Unfortunately, even the existence of a solution may not be guaranteed. A famous example is furnished by Lewy's linear equation without solutions, regardless of the type or form of data prescribed ([6]). Abundant instances in which the uniqueness of a solution or its continuous dependence is violated can be found (e.g., [3], [7], [8]).

Once these questions are settled, an additional one may be posed for a time dependent problem: Does the solution exist for all time? In illustration, we present the initial value problem

$$\frac{dy}{dt} = \frac{1}{1-y}, \quad t > 0; \quad y(0) = 0,$$

whose solution,  $y(t) = 1 - \sqrt{1-2t}$ , cannot be continued past  $t = 1/2$ , since when  $y$  reaches one,  $dy/dt$  becomes unbounded.

Furthermore, for an evolution equation with a global solution, another consideration can be made: Are there any stationary solutions, i.e. solutions independent

of time, and are any of these steady states stable in the sense that when a solution starts “near” a steady state, it stays “near” that steady state for all time?

To answer the questions just posed is sometimes very difficult. To rise to the challenge is exciting, however. For this reason, we will conduct a series of discussions on these topics.

In this thesis, we consider three nonlinear initial-boundary value problems of parabolic type. Our main interest in these problems is twofold. First, we focus on a problem related to the so called “quenching” phenomenon. (As time tends to an instant, the solution remains bounded, while its derivative becomes unbounded.) Since Kawarada [4] first proposed the problem formulated from a physical model in 1975, a lot of authors have extended some results to more general problems. However, problems such as the location of quenching points or the behavior of solutions at quenching were not completely understood. Therefore, in this direction, we undertake a study in more detail.

Second, we investigate the long time behavior of solutions of Burgers’ equation with nonlocal boundary conditions. There are two reasons for considering these problems. One arises from the physical motivation, because some boundary conditions represent a forcing of the flux. The other comes from a purely mathematical point of view, since to the best of our knowledge, no one has done nonlocal problems of the type considered here. Although the presence of the nonlocal term makes discussion more complicated, all results for local boundary conditions in a previous work of Levine [5] are preserved.

### **Explanation of Dissertation Format**

The dissertation is written in the alternate dissertation format. Each part represents a paper which has been submitted to a scholarly journal for publication.

## PART I. ON THE BLOW UP OF $u_t$ AT QUENCHING

### Abstract

Let  $\Omega$  be a bounded convex domain in  $\mathbf{R}^n$  with smooth boundary. We consider the problems (C):  $u_t = \Delta u + \varphi(u)$  in  $\Omega \times (0, T)$ , while  $u = 0$  on  $\partial\Omega \times (0, T)$  and  $u(x, 0) = u_0(x)$ . Here  $\varphi(u) : (-\infty, A) \rightarrow (0, \infty)$  ( $A > 0$ ) satisfies  $\varphi'(u) \geq 0$ ,  $\varphi''(u) \geq 0$ , and  $\lim_{u \rightarrow A^-} \varphi(u) = +\infty$ , while  $u_0$  satisfies  $\Delta u_0(x) + \varphi(u_0(x)) \geq 0$ . We show that if  $u$  quenches (reaches  $A$  in finite time), then the quenching points are in a compact subset of  $\Omega$  and  $u_t$  blows up. We also extend the result to the third boundary value problem.

### 1. Introduction

In this paper [10], Kawarada studied the following initial boundary value problem:

$$\begin{aligned}
 (A) \quad & u_t = u_{xx} + \frac{1}{1-u}, & 0 < x < L, & \quad t > 0, \\
 & u(0, t) = u(L, t) = 0, & & \quad t > 0, \\
 & u(x, 0) = 0, & 0 \leq x \leq L.
 \end{aligned}$$

He showed the following:

- (i) If  $L > 2\sqrt{2}$ , then  $u\left(\frac{L}{2}, t\right)$  reaches one in finite time.
- (ii) If  $u\left(\frac{L}{2}, t\right)$  reaches one in finite time, then  $u_t\left(\frac{L}{2}, t\right)$  becomes unbounded in finite time.

When (ii) occurs, Kawarada says that  $u$  quenches in finite time. Unfortunately, his methods do not appear to extend readily to more general cases. Therefore a weaker definition was posed in [2],[11] where  $u$  is said to quench if (i) occurs. For more general problems of parabolic type, some results were obtained over the last few years by several authors in [2],[3],[11],[12],[13],[14].

In [4] the authors remark that Kawarada's proof of (ii) is incomplete. They give a complete proof of (ii) using elementary arguments for a more general class of nonlinearities which includes those of the form  $(1 - u)^{-\beta}$  for  $\beta \geq 1$ .

In [8], the author has shown that if  $\beta \geq 3$ , then the behavior of  $u$  at a quenching point is asymptotically precisely the same as that for the solution of the initial value problem  $y' = (1 - y)^{-\beta}$ ,  $y(0) = 0$  on  $(0, T)$ . In [9], he extended these results to radial solutions in sufficiently large balls.

Recently, A. Acker and B. Kawohl in [1] investigated an analogous problem in several dimensions:

$$(B) \quad \begin{aligned} u_t &= \Delta u + f(u), & x &\in B_a, & t &> 0, \\ u &= 0, & x &\in \partial B_a, & t &> 0, \\ u(x, 0) &= u_0(x), & x &\in B_a, \end{aligned}$$

where  $B_a$  is a ball in  $\mathbf{R}^n$  with center at the origin. Under the assumptions that  $u = u(r, t)$  where  $r = |x|$  and the initial values satisfy  $u(r, 0) \geq 0$ ,  $u_r(r, 0) \leq 0$ ,  $u_t(r, 0) \geq 0$  and  $u_{rt}(r, 0) \leq 0$  and with some mild restrictions on the nonlinearity  $f(u)$ , they proved that if  $u$  quenches, then the only quenching point is the origin and  $u_t(0, t)$  is unbounded.

In this paper, we consider the more general problem:

$$(C) \quad \begin{aligned} u_t &= \Delta u + \varphi(u), & x &\in \Omega, & 0 < t < T, \\ u &= 0, & x &\in \partial\Omega, & 0 < t < T, \\ u(x, 0) &= u_0(x), & x &\in \Omega. \end{aligned}$$

Here  $\Omega$  is a bounded convex domain in  $\mathbf{R}^n$  with smooth boundary.  $\varphi(u) : (-\infty, A) \rightarrow (0, \infty)$  ( $A > 0$ ) satisfies  $\varphi'(u) \geq 0$ ,  $\varphi''(u) \geq 0$ , and  $\lim_{u \rightarrow A^-} \varphi(u) = +\infty$ ; while the initial datum satisfies  $0 \leq u_0 < A$  and  $\Delta u_0 + \varphi(u_0) \geq 0$ .

In §2, we prove that if  $u$  quenches in finite or infinite time, then the quenching points are in a compact set. In §3, we show that  $u_t$  blows up at finite quenching time. In §4, we extend the result in §3 to the third boundary value problem. Our arguments are based on modifications of those in [6] for blow up problems.

Our result also allows us to obtain blow up results. By means of transformation  $v = -\ln(1 - u)$ , the differential equation in (C) becomes

$$v_t = \Delta v + e^v \varphi(1 - e^{-v}) - |\nabla v|^2.$$

Thus, as remarked in [1], blow up of solutions of this equation is equivalent to the quenching of solutions (C). Thus, following [1], we may say that the set of blow-up points for the Dirichlet initial-boundary value problem for the above equation be in a compact subset of  $\Omega$  and whenever  $v$  blows up in finite time so does  $e^{-v}v_t$ .

In [5], the authors considered the Dirichlet initial boundary value problem for

$$v_t = \Delta v - |\nabla v|^q + |v|^{p-1}v$$

and showed that if  $1 < q \leq 2p/(p+1)$ ,  $p > 1$ , these solutions blow up in finite time. Clearly the case  $q = 2$  is excluded from their result. On the other hand, if we consider the equation

$$v_t = \Delta v + \varepsilon e^{(\beta+1)v} - |\nabla v|^2,$$

then, with  $\varphi(u) = \varepsilon(1 - u)^{-\beta}$ , our results tell us that for  $\beta > 0$ , some solutions  $v$  (with  $v_t$ ) blow up if  $\varepsilon$  is sufficiently large. (Blow up results are well known for these last two equations when  $|\nabla v|^2$  is not present on the right-hand side.)

## 2. The Location of the Quenching Points

We begin with an introduction of some remarks and definitions. We show that the quenching points lie in a compact subset of  $\Omega$ .

Let  $\nu$  be a unit vector in  $\mathbf{R}^n$  and let  $T_\lambda$  be the hyperplane  $\nu \cdot x = \lambda$ . Let the plane move continuously toward  $\Omega$  with the same normal, i.e., decrease  $\lambda$ , until it begins to intersect  $\bar{\Omega}$ . From that moment on, at every stage the plane  $T_\lambda$  will cut off from  $\Omega$  an open cap  $\Sigma(\lambda)$  associated with  $\nu$ . Note that  $\nu$  is the outer normal of  $\partial\Omega$  at the point  $p$  of the boundary which  $T_\lambda$  first touches. For simplicity, we use  $\lambda$  to denote the distance from  $p$  to  $T_\lambda$ , and denote  $\Omega \times \{t = \eta\}$  by  $\Omega_\eta$  ( $0 < \eta < T$ ).

**Lemma 2.1.** *If  $\varphi(u) : (-\infty, A) \rightarrow (0, \infty)$  is continuously differentiable and  $u_0(x) \geq 0$ , then for every  $\eta$  ( $0 < \eta < T$ ) at every point  $p_0$  on  $\partial\Omega_\eta \times (\eta, T)$ , there is a cap  $\Sigma(\lambda_0)$ , such that  $\frac{\partial u}{\partial n_{p_0}} < 0$  for any  $(x, t) \in \Sigma(\lambda_0) \times (\eta, T)$ .*

**PROOF:** Since  $u > 0$  in  $\Omega_\eta \times [\eta, T)$  and  $u = 0$  on  $\partial\Omega_\eta \times (\eta, T)$ , this follows from the similar argument in [7].

Now we call  $D(\lambda_0) = \Sigma(\lambda_0) \times (\eta, T)$  the cylinder.

For fixed  $\eta \in (0, T)$ ,  $\partial\Omega_\eta$  is a compact set. Hence  $\lambda_{\max} = \max\{\lambda_0\}$  and  $\lambda_{\min} = \min\{\lambda_0\}$  exist, and  $\lambda_{\max} \geq \lambda_{\min} > 0$ . Let  $\Omega_{\lambda_0} = \{p \mid p_{\lambda_0} \bar{p} \cdot p_{\lambda_0} \bar{p}_0 < 0\}$ , with  $p_0$  on  $\partial\Omega_\eta$  and  $p_{\lambda_0}$  on  $T_{\lambda_0}$  with  $p_{\lambda_0} \bar{p}_0$  perpendicular to  $T_{\lambda_0}$ .

Let  $\Omega'_\eta = \bigcap_{p_{\lambda_0} \in T_{\lambda_0}} \Omega_{\lambda_0}$ .  $\Omega'_\eta$  is the complement of the union of all  $\Sigma(\lambda_0)$ 's with respect to  $\Omega_\eta$ . Clearly  $\Omega'_\eta \subset \subset \Omega_\eta$ .

Now we state our main result as follows:

**Theorem 2.2.** Assume that the conditions in Lemma 2.1 hold, and  $\varphi'(u) \geq 0$  for  $0 \leq u < A$ ,  $0 \leq u_0(x) < A$ . Then the quenching points are in a compact subset of  $\Omega$ .

PROOF: For any point  $p_*(x^*, t^*) \in (\Omega_\eta \setminus \Omega'_\eta) \times (\eta, T)$ , there is a point  $p_0(x^0, t^*) \in \partial\Omega \times \{t = t^*\}$  such that the line  $L_0$  through  $p_0$  and  $p_*$  has the same direction as  $n_{p_0}$  at  $p_0$ . By Lemma 2.1,  $p_*$  is contained in a cylinder  $D(\lambda_0)$ . We construct a new cylinder  $D(\lambda_*)$  by using  $T_{\lambda_*}$  instead of  $T_{\lambda_0}$ , where  $0 < \lambda_* < \lambda_0$  is such that  $p_*$  remains in  $D(\lambda_*)$ . From the conclusion of Lemma 2.1, we know that  $\frac{\partial u}{\partial n_{p_0}} < 0$  in  $D(\lambda_*)$ ; in particular for all  $(x, t)$  on  $T_{\lambda_*} \cap (\Omega \times (\eta, T))$ . Without loss of generality, we may assume  $n_{p_0} = (1, 0, \dots, 0)$ ,  $x = (x_1, x')$ , ( $x' = (x_2, \dots, x_n)$ ), and  $(x, t^*)$  is on the line  $L_0$ . We let  $p_0 = (x_1^0, x', t^*)$ ,  $p_* = (x_1^*, x', t^*)$ , and the point  $\bar{p} = (\bar{x}_1, x', t^*)$  where the line  $L_0$  intersects  $T_{\lambda_*}$ , i.e.,  $T_{\lambda_*} = \{x_1 = \bar{x}_1\}$ . Obviously  $\frac{\partial u}{\partial n_{p_0}} = \frac{\partial u}{\partial x_1}$ .

Now we define a function in  $D(\lambda_*)$  by:

$$F(x, t) = \frac{\partial u}{\partial x_1} + c(x_1 - \bar{x}_1)$$

Here  $c$  is a positive constant to be determined. We have

$$F_t = u_{x_1 t}$$

$$\Delta F = \Delta u_{x_1}$$

so  $F_t - \Delta F = \varphi'(u)u_{x_1} \leq 0$  in  $D(\lambda_*)$ . On the boundary of  $D(\lambda_*)$ , we have at  $T_{\lambda_*} = \{x_1 = \bar{x}_1\}$ ,  $F = u_{x_1} < 0$ , on  $D(\lambda_*) \cap \{t = \eta\}$ ,

$$F = u_{x_1}(x, \eta) + c(x_1 - \bar{x}_1) \leq \max_{D(\lambda_*) \cap \{t = \eta\}} u_{x_1}(x, \eta) + c\lambda_{\max} < 0$$

provided  $c < \frac{-\max_{D(\lambda_*) \cap \{t = \eta\}} u_{x_1}(x, \eta)}{\lambda_{\max}}$ .

To show that  $F < 0$  on  $\partial D(\lambda_*) \cap (\partial\Omega \times (\eta, T))$ , we consider the problem:

$$\begin{aligned} v_t &= \Delta v + \varphi(0), & x &\in \Omega, & 0 < t < T, \\ v &= 0, & x &\in \partial\Omega, & 0 < t < T, \\ v(x, 0) &= 0, & x &\in \Omega. \end{aligned}$$



Since  $\varphi'(u) \geq 0$ , we have  $u \geq v$  by the maximum principle, and  $u \not\equiv v$ . It follows from the Hopf-Weinberger maximum principle that

$$\frac{\partial u}{\partial n} < \frac{\partial v}{\partial n} \leq -c_1 < 0 \quad \text{on } \partial\Omega \times (0, T).$$

In particular

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial n} \cos\langle n, x_1 \rangle \leq -c_1 \cos\langle n, x_1 \rangle \quad \text{on } \partial D(\lambda_*) \cap (\partial\Omega \times (\eta, T)).$$

We may assume

$$\cos\langle n, x_1 \rangle \geq c_2 > 0$$

for some  $c_2 > 0$  and every point  $(x, t) \in \partial D(\lambda_*) \cap (\partial\Omega \times (\eta, T))$ .

Therefore on  $\partial D(\lambda_*) \cap (\partial\Omega \times (\eta, T))$ ,

$$\frac{\partial u}{\partial x_1} + c(x_1 - \bar{x}_1) \leq -c_1 c_2 + c \lambda_{\max} < 0 \quad \text{if } c < \frac{c_1 c_2}{\lambda_{\max}}.$$

Letting  $c = \min \{-\max u_{x_1}(x, \eta)/\lambda_{\max}, c_1 c_2/\lambda_{\max}\}$ , we find that

$$F(x, t) \leq 0 \quad \text{in } D(\lambda_*),$$

or

$$-u_{x_1} \geq c(x_1 - \bar{x}_1).$$

Thus for any point  $p_* = (x_1^*, x', t^*)$ , we see that

$$\begin{aligned} \int_{\bar{x}_1}^{x_1^*} (-u_{x_1}) dx_1 &\geq c \int_{\bar{x}_1}^{x_1^*} (x_1 - \bar{x}_1) dx_1 \\ u(\bar{x}_1, x', t^*) - u(x_1^*, x', t^*) &\geq \frac{c}{2} (x_1^* - \bar{x}_1)^2 \end{aligned}$$

or

$$\begin{aligned}
u(x_1^*, x', t^*) &\leq u(\bar{x}_1, x', t^*) - \frac{c}{2}(x_1^* - \bar{x}_1)^2 \\
&\leq A - \frac{c}{2}(x_1^* - \bar{x}_1)^2.
\end{aligned}$$

Since  $p_*(x^*, t^*)$  is an arbitrary point in  $(\Omega_\eta \setminus \Omega'_\eta) \times (\eta, T)$ , the set of quenching points lies in a compact subset of  $\Omega$ .

REMARK 2.1: In Theorem 3.3 of [6], the authors define (with  $J$  instead of  $F$ )

$$F(x, t) = u_{x_1} + c(x_1 - \bar{x}_1)\mathcal{F}(u)$$

where  $\mathcal{F}$  is required to satisfy (2.23) of [6], i.e.

$$\int_0^\infty [\mathcal{F}(s)]^{-1} ds < \infty$$

among other conditions. Our  $\mathcal{F}(\equiv 1)$  does not satisfy this condition.

REMARK 2.2: We do not require that  $T$  be finite in the proof of the theorem and the result thus holds for  $0 < T \leq \infty$ .

REMARK 2.3: If  $u_0(x)$  satisfies the condition (2.2) in [7] on a part of  $\Omega$ , then we can locate the quenching points more precisely. Especially, if  $\Omega$  is a ball in  $\mathbb{R}^n$  and  $u$  is a radial solution with  $\frac{\partial u_0}{\partial r} \leq 0$ , then the center of the ball is the only quenching point.

REMARK 2.4: If in (C) we replace  $\varphi$  by  $\varepsilon\varphi$  where  $\varepsilon > 0$ , then it is known that for all sufficiently large  $\varepsilon$ , the set of quenching points is not empty [2],[3],[11],[14].

### 3. The Blow Up of $u_t$ at Quenching

As an application of Theorem 2.2, we now show that when  $u$  quenches, then  $u_t$  blows up. Here we use a modification of an argument of [6].

**Theorem 3.1.** Assume, in addition to the hypotheses in Theorem 2.2, that  $\varphi''(u) \geq 0$  for  $0 \leq u < A$  and  $\Delta u_0 + \varphi(u_0) \geq 0$  in  $\Omega$ . Then if  $u$  quenches in finite time,  $u_t$  blows up.

PROOF: Let  $\Omega''_\eta$  be the set  $\{x \mid \text{dist}(x, \Omega'_\eta) \leq \frac{1}{2} \lambda_{\min}\}$ , it is clear that  $\Omega'_\eta \subset \subset \Omega''_\eta \subset \subset \Omega_\eta$ .

Consider the function  $G(x, t) = u_t - \delta\varphi(u)$  in  $\Omega''_\eta \times (\eta, T)$ , where  $\delta$  is an undetermined positive constant.

$$\begin{aligned} G_t &= u_{tt} - \delta\varphi'(u)u_t, \\ \Delta G &= \Delta u_t - \delta\varphi''(u)|\nabla u|^2 - \delta\varphi'(u)\Delta u, \\ G_t - \Delta G &= \varphi'(u)u_t + \delta\varphi''(u)|\nabla u|^2 - \delta\varphi'(u)\varphi(u) \\ &= \varphi'(u)(G + \delta\varphi(u)) + \delta\varphi''(u)|\nabla u|^2 - \delta\varphi'(u)\varphi(u) \\ &= \varphi'(u)G + \delta\varphi''(u)|\nabla u|^2, \end{aligned}$$

so

$$G_t - \Delta G - \varphi'(u)G \geq 0.$$

From the maximum principle, it follows that  $G$  cannot take negative minimum in  $\Omega''_\eta \times (\eta, T)$ .

On the parabolic boundary,  $\varphi(u) \leq c$  in  $\Omega''_\eta$ , while on  $\partial\Omega''_\eta \times (\eta, T)$  (by Theorem 2.2),  $\varphi(u) \leq c$  also. On the other hand, using the condition  $\Delta u_0 + \varphi(u_0) \geq 0$  in  $\Omega$ , we see that  $u_t > 0$  in  $\Omega \times (0, T)$ . Thus  $u_t \geq c_1 > 0$  for  $(x, t)$  on  $\partial\Omega''_\eta \times (\eta, T)$  and  $(x, \eta)$  in  $\Omega'_\eta$ . Hence if  $\delta < \frac{c_1}{c}$ , then  $G > 0$  in  $\Omega''_\eta$ . It follows that  $G \geq 0$  in  $\Omega''_\eta \times (\eta, T)$ , i.e.  $u_t \geq \delta\varphi(u)$ . Thus  $\lim_{u \rightarrow A^-} u_t = +\infty$ .

REMARK: In the proof of the theorem,  $c_1$  depends on  $T$ . This means that the method does not apply to the case  $T = \infty$ . This is in agreement with the observation

in [11] where it was shown that when  $\Omega$  is an interval and when  $\lim_{t \rightarrow \infty} u(x_0, t) = A$ , then  $\lim_{t \rightarrow \infty} u_t(x_0, t) = 0$ .

#### 4. The Blow Up of $u_t$ for the Robin Condition

In this section, we extend some of the previous results to the following problem:

$$(D) \quad \begin{aligned} u_t &= \Delta u + \varphi(u), & x &\in \Omega, & 0 < t < T, \\ \frac{\partial u}{\partial n} + \beta u &= 0, & x &\in \partial\Omega, & 0 < t < T, \\ u(x, 0) &= u_0(x), & x &\in \Omega, \end{aligned}$$

where  $\beta = \beta(x, t) > 0$ ,  $\beta_t \leq 0$ .

Because the main theorem in §2 is based on Lemma 2.1 which strongly depends on the Dirichlet problem (C), it cannot be applied to problem (D). Instead, we use other arguments to achieve the same result as in Theorem 3.1.

**Lemma 4.1.** *If  $\varphi(u) > 0$ ,  $\varphi''(u) \geq 0$  for  $0 \leq u < A$ , and  $\lim_{u \rightarrow A^-} \varphi(u) = +\infty$ , then there exists a number  $A_0$  ( $0 < A_0 < A$ ), such that  $u\varphi'(u) \geq \varphi(u)$  if  $u \geq A_0$ .*

PROOF: Let  $\Phi(u) = \frac{u}{\varphi(u)}$ . We find that  $\Phi(0) = 0$ ,  $\Phi(u) > 0$  for  $0 < u < A$ , and  $\lim_{u \rightarrow A^-} \Phi(u) = 0$ .

Thus setting  $\Phi(A) = 0$ , we have that  $\Phi(u)$  is continuous in  $[0, A]$ , and differentiable in  $(0, A)$ . Let  $A_0$  be the point nearest  $A$  where  $\Phi(u)$  obtains its maximum, then  $\Phi'(A_0) = 0$  and  $\Phi'(u) \leq 0$ , or  $u\varphi'(u) \geq \varphi(u)$  for  $u \geq A_0$ .

**Lemma 4.2.** *If the conditions in Lemma 4.1 are satisfied,  $\varphi'(u) \geq 0$  for  $0 \leq u < A$  and  $\Delta u_0 + \varphi(u_0) \geq 0$  in  $\Omega$ , then if  $u$  is the solution of the problem (D),  $u_t \geq c > 0$  for  $(x, t) \in \Omega \times (\eta, T)$ .*

PROOF: Set  $v(x, t) = u_t(x, t)$ . Then  $v(x, t)$  satisfies

$$\begin{aligned} v_t &= \Delta v + \varphi'(u)v, & x \in \Omega, & \quad 0 < t < T, \\ \frac{\partial v}{\partial n} + \beta v &\geq 0, & x \in \partial\Omega, & \quad 0 < t < T, \\ v(x, 0) &= \Delta u_0 + \varphi(u_0) \geq 0, & x \in \Omega. \end{aligned}$$

By the maximum principle,  $v > 0$  in  $\Omega \times (0, T)$ . Since  $\varphi'(u) \geq 0$ ,

$$v_t \geq \Delta v \quad \text{in } \Omega \times (0, T).$$

Now consider the related problem:

$$\begin{aligned} w_t &= \Delta w, & x \in \Omega, & \quad \eta < t < T, \\ \frac{\partial w}{\partial n} + \beta w &= 0, & x \in \partial\Omega, & \quad \eta < t < T, \\ w(x, \eta) &= v(x, \eta), & x \in \Omega. \end{aligned}$$

Since  $v(x, t) \geq w(x, t)$  and  $w(x, t) \geq c$  in  $\Omega \times (\eta, T)$ , the conclusion follows.

Now we can use the technique adopted in [6] to show the following:

**Theorem 4.2.** *Under the same hypotheses as in Theorem 3.1, we have that  $u_t$  blows up at quenching.*

PROOF: Set  $G(x, t) = u_t - \delta\varphi(u)$ .

As before

$$G_t - \Delta G - \varphi'(u)G \geq 0 \quad \text{in } \Omega \times (\eta, T).$$

By applying Lemma 4.2,  $G(x, \eta) > 0$  for  $x \in \Omega$  if  $\delta < \frac{c}{\max_{\Omega_\eta} \varphi(u(x, \eta))}$ . Then we claim that  $G(x, t) \geq 0$  for all  $(x, t) \in \partial\Omega \times (\eta, T)$  provided  $\delta \leq \frac{c}{\varphi(A_0)}$ .

Assume the contrary:  $G$  takes a negative value at some point  $(x_0, t_0) \in \partial\Omega \times (\eta, T)$ .

Let  $G(x^*, t^*)$  be a negative minimum on  $\partial\Omega \times [\eta, t_0]$ . From the choice of  $\delta$  and the fact that  $G$  cannot have negative minimum in  $\Omega \times [\eta, t_0]$ , it is clear that  $u(x^*, t^*) > A_0$ , and that

$$\frac{\partial G}{\partial n} + \beta G < 0 \quad \text{at } (x^*, t^*)$$

or

$$\left( \frac{\partial u_t}{\partial n} + \beta u_t \right) - \delta \left( \frac{\partial \varphi}{\partial n} + \beta \varphi \right) < 0.$$

It follows that

$$\frac{\partial \varphi}{\partial n} + \beta \varphi > 0 \quad \text{at } (x^*, t^*).$$

Note that

$$\frac{\partial \varphi}{\partial n} = \varphi'(u) \frac{\partial u}{\partial n} = -\beta u \varphi'(u),$$

we find that  $\varphi(u) - u\varphi'(u) > 0$  for  $u = u(x^*, t^*) > A_0$ .

This leads to a contradiction.

Therefore we have shown that  $G(x, t) \geq 0$  in  $\Omega \times (\eta, T)$ , which yields the desired result.

## 5. References

- [1] A. Acker and B. Kawohl. *Remarks on quenching*. Nonlinear Anal. TMA (in press).
- [2] A. Acker and W. Walter. *The quenching problem for nonlinear parabolic equations*. Lecture Notes in Mathematics 564. Springer-Verlag, New York, 1976.
- [3] A. Acker and W. Walter. *On the global existence of solutions of parabolic differential equations with a singular nonlinear term*. Nonlinear Anal. 2(1978), 449–505.
- [4] C. Y. Chan and M. K. Kwong. *Quenching phenomena for singular nonlinear parabolic equations*. Nonlinear Anal. TMA 12(1988), 1377–1383.
- [5] M. Chipot and F. B. Weissler. *Some blow up results for a nonlinear parabolic equation with a gradient term*. IMA Preprint Series #298.
- [6] A. Friedman and B. McLeod. *Blow-up of positive solutions of semilinear heat equations*. Indiana Univ. Math. J. 34(1985), 425–447.
- [7] B. Gidas, W.-M. Ni and L. Nirenberg. *Symmetry and related properties via the maximum principle*. Commun. Math. Phys. 68(1979), 209–243.
- [8] J. Guo. *On the quenching behavior of the solution of a semilinear parabolic equation*. J. Math. Anal. Appl. (in print).
- [9] J. Guo. *On the semilinear elliptic equation,  $\Delta w - \frac{1}{2}y \cdot \nabla w + \lambda w - w^{-\beta} = 0$  in  $\mathbb{R}^n$* , (in print).
- [10] H. Kawarada. *On the solutions of initial-boundary value problems for  $u_t = u_{xx} + \frac{1}{1-u}$* . Publ. RIMS Kyoto Univ. 10(1975), 729–736.
- [11] H. A. Levine and J. T. Montgomery. *The quenching of solutions of some nonlinear parabolic equations*. SIAM J. Math. Anal. 11(1980), 842–847.
- [12] H. A. Levine. *The quenching of solutions of linear parabolic and hyperbolic equations with nonlinear boundary conditions*. SIAM J. Math. Anal. 14(1983), 1139–1153.
- [13] H. A. Levine. *The phenomenon of quenching: a survey*. in Trends in the Theory and Practice of Non-linear Analysis. Ed. V. Lakshmikantham. Elsevier Science Publ., North Holland, 1985, 275–286.
- [14] H. A. Levine. *Quenching, nonquenching, and beyond quenching for solutions of some parabolic equations*. Ann. Mat. Pura Appl. (IV) 155(1989), 243–260.

## PART II. QUENCHING FOR SOLUTIONS OF A PLASMA TYPE EQUATION

### Abstract

In this paper, we examine the first initial boundary value problem for the fast diffusion equation  $u_t = (u^m)_{xx} + \varepsilon(1 - u)^{-\alpha}$ ,  $0 < m < 1$ ,  $\varepsilon, \alpha > 0$  on  $(0, 1) \times (0, T)$ . We characterize the set of stationary solutions and show that quenching can occur for solutions with certain initial data. We also extend some results to the porous medium problem.

### 1. Introduction

In this paper, we investigate the global existence and nonexistence of solutions of the first initial boundary value problem

$$\begin{aligned}
 (I) \quad & u_t = (u^m)_{xx} + \varepsilon(1 - u)^{-\alpha} & 0 < x < 1, \quad t > 0, \\
 & u(0, t) = u(1, t) = 0 & t > 0, \\
 & u(x, 0) = u_0(x) & 0 \leq x \leq 1.
 \end{aligned}$$

Here  $0 < m < 1$ ,  $\varepsilon, \alpha$  are positive, and  $0 \leq u_0(x) < 1$ . This problem arises in the area of plasma physics, where  $u$  denotes plasma density, so it is natural to restrict  $u$  to be nonnegative. Since the diffusion coefficient,  $mu^{m-1}$ , tends to infinity as  $u \rightarrow 0$ ; we sometimes call this case, the “fast diffusion” case.

Following a popular definition, we say that the solution of (I) quenches if it reaches one in finite time. The case  $m = 1$  has been investigated by many authors. For



details, see [3]. The exploration of the current problem, however, has not been made yet, and hence we focus on it. Some results will also be mentioned regarding the porous medium problem ( $m > 1$ ), the so called “slow diffusion” case. The analysis will be carried out in the manner of [5], in which Levine gave a complete qualitative study of problem (I) with  $m = 1$ .

The plan of our paper is as follows: In the next section, we present the comparison theorem and local existence of solutions for (I). In the third section, we characterize the set of stationary solutions of (I). Finally, in the last section, we establish stability and quenching results for problem (I).

## 2. Comparison and Local Existence

At the beginning of this section we proceed to define subsolution, supersolution, and solution of (I).

For convenience, let  $\Omega = (0, 1)$ ,  $D_T = \Omega \times (0, T)$ , and  $f(u) = (1 - u)^{-\alpha}$ .

**DEFINITION:** A *solution*  $u$  of problem (I) on  $[0, T]$  is a function  $u$  with the following properties:

- (i)  $u \in C([0, T]; L^1(\Omega)) \cap L^\infty(D_T)$ ;
- (ii)  $u$  satisfies

$$\begin{aligned} \int_{\Omega} u(T)\varphi(T) - \int \int_{D_T} (u\varphi_t + u^m\varphi_{xx}) \\ = \int_{\Omega} u_0\varphi(0) + \varepsilon \int \int_{D_T} f(u)\varphi \end{aligned}$$

for  $0 \leq t \leq T$  and  $\varphi \in C^{2,1}(\overline{D_T})$ ,  $\varphi \geq 0$  on  $D_T$ ,  $\varphi = 0$  at  $x = 0$  and  $x = 1$ .

A *subsolution* (*supersolution*) of problem (I) is defined by (i) and (ii) with equality replaced by  $\leq$  ( $\geq$ ).

In what follows, we shall exhibit a powerful tool for this analysis — the comparison principle, which will be used later.

**THEOREM 2.1.** *Let  $u$  ( $< 1$ ) be a subsolution and  $v$  ( $< 1$ ) be a supersolution of problem (I) with initial data  $u_0$  and  $v_0$ , respectively. Then if  $u_0 \leq v_0$ , it follows that  $u \leq v$  on  $D_T$ .*

**PROOF:** In Theorem 2.4 of [4], the author built up a comparison principle for problem (I) with nonlinearity  $f(u)$  being “locally Lipschitz” continuous. If both subsolution and supersolution are bounded away from 1, then the condition is satisfied, and the conclusion follows.

By means of the comparison theorem, we can prove the local existence of solutions of (I).

**THEOREM 2.2.** *For  $0 \leq u_0 < 1$ , there exists a  $T_0 < \infty$  such that problem (I) has a unique nonnegative solution on  $D_{T_0}$ .*

**PROOF:** Clearly  $u \equiv 0$  is a subsolution. If we can find a supersolution  $v$  with  $v_0 \geq u_0$  such that for some  $T_0 < \infty$ ,  $v < 1$  on  $D_{T_0}$ , then by comparison, we know that whenever  $u$  is a solution of (I), it should be bounded away from 1. Making use of Theorem 3.1 in [4] or Theorem 4.1 of [6] for  $0 < m < 1$ , we obtain the desired conclusion.

To this end, we let  $v = v(t)$  be a solution of the ordinary differential equation

$$\frac{dv}{dt} = \frac{\varepsilon}{(1-v)^\alpha}, \quad v(0) = \|u_0\|_{L^\infty(\Omega)}.$$

Solving it, we obtain that  $v = 1 - \left[ (1 - v(0))^{\alpha+1} - \varepsilon(\alpha+1)t \right]^{\frac{1}{\alpha+1}}$ . For sufficiently small  $t$ , the quantity in the bracket can be kept positive, hence  $v < 1$  and  $v$  is our desired supersolution.

With the help of the comparison theorem, we can also show the monotonicity of solutions of (I) in time.

**THEOREM 2.3.** *If the initial datum  $u_0 \in C^2(\Omega)$  satisfies  $(u_0^m)'' + \varepsilon f(u_0) \geq 0$  ( $\leq 0$ ), then  $u(x, t)$  is monotonically increasing (decreasing) in  $t$ .*

**PROOF:** We prove the first assertion only because the second statement can be shown by the same argument.

Obviously,  $u_0$  is a subsolution of (I). Recalling the comparison theory, we have that  $u_0 \leq u$ . Let  $w(x, t) = u(x, t + h)$  ( $h > 0$  arbitrary) to find that  $w(x, 0) = u(x, h) \geq u_0(x)$ ,  $w(x, t)$  is a solution, and thus a supersolution of (I). Applying Theorem 2.1 again yields that  $u(x, t + h) \geq u(x, t)$  for any  $t > 0$ .

**REMARK 2.1:** Without any difficulty, all results in this section can be extended into  $n$  dimensional space ( $n > 1$ ) with more general nonlinearity  $f(u)$ . They can also apply to the porous medium problem.

**REMARK 2.2:** The solutions which we discuss possess much stronger regularity properties than those required by the definition of a solution. For the purpose of this paper, however, we need discuss only weaker solutions.

### 3. Stationary Solutions

We begin with the definition of stationary solutions of problem (I). We are interested only in nonnegative solutions. A function  $v: [0, 1] \rightarrow [0, a]$  ( $a \leq 1$ ) is a stationary solution of (I) when it satisfies the following:

$$(II) \quad \begin{aligned} (v^m)'' + \varepsilon f(v) &= 0 & 0 < x < 1, \\ v(0) &= v(1) = 0, \end{aligned}$$

in the sense that  $v \in C(\overline{\Omega})$  and  $\int_{\Omega} (\varphi'' v^m + \varepsilon \varphi f(v)) dx = 0$  for all  $\varphi \in C^2(\overline{\Omega})$ ,  $\varphi \geq 0$  and  $\varphi(0) = \varphi(1) = 0$ .

To obtain a complete bifurcation analysis, we need more regularity than those in the (weak) definition. For this reason, we first restrict the stationary solution within a limit:  $v < 1$ . Since  $f$  is locally Lipschitz, the equation in (II) holds classically, i.e.,  $v \in C^2(\Omega)$ ,  $v(0) = v(1) = 0$ .

Now suppose that  $v$  is a positive classical solution of (II). Then  $v(x)$  has a maximum at  $\xi \in (0, 1)$  and it follows that  $v'(\xi) = 0$ . Conversely, for  $\xi$  under certain conditions and for  $0 < \mu < 1$ , consider the problem

$$(II') \quad \begin{aligned} (v^m)'' + \varepsilon f(v) &= 0, \\ v(\xi) &= \mu, \quad v'(\xi) = 0, \end{aligned}$$

then  $v$  is also a positive solution of problem (II).

Therefore, in order to discuss solutions of (II), we may solve (II') first. Integrating the equation in (II') multiplied by  $(v^m)'$ , and using the initial conditions, we arrive at

$$(3.1) \quad \frac{1}{2} (v^m)'^2 + \varepsilon m F(v) = \varepsilon m F(\mu),$$

here  $F(v) = \int_0^v s^{m-1} f(s) ds$  is an incomplete Beta function.

Integrate (3.1) to obtain

$$(3.2) \quad \sqrt{\frac{m}{2}} \int_v^\mu \frac{\eta^{m-1}}{\sqrt{F(\mu) - F(\eta)}} d\eta = \sqrt{\varepsilon} |\xi - x|.$$

For a positive solution of (II),  $v = 0$  only at two endpoints. Thus, from

$$(3.3) \quad \sqrt{\frac{m}{2}} \int_0^\mu \frac{\eta^{m-1}}{\sqrt{F(\mu) - F(\eta)}} d\eta = \sqrt{\varepsilon} \xi = \sqrt{\varepsilon} (1 - \xi),$$

it follows that  $\xi = \frac{1}{2}$ .

Let  $G(\mu) = \sqrt{2m} \int_0^\mu \frac{\eta^{m-1}}{\sqrt{F(\mu) - F(\eta)}} d\eta$ . Then (3.3) is equivalent to

$$(3.4) \quad G(\mu) = \sqrt{\varepsilon}.$$

Note that  $v(1-x)$  is also a solution of (II). Combination of this fact with  $v(\frac{1}{2}) = \mu$  yields that for  $0 \leq x \leq \frac{1}{2}$ ,  $v(x)$  is implicitly given by

$$(3.5) \quad \sqrt{\frac{m}{2}} \int_0^{v(x)} \frac{\eta^{m-1}}{\sqrt{F(\mu) - F(\eta)}} d\eta = \sqrt{\varepsilon} x,$$

and by  $v(x) = v(1-x)$ , if  $\frac{1}{2} < x < 1$ , with  $\mu$  satisfying (3.4). Therefore, to achieve the aim of this section, we only need focus our attention on the solution of (3.4).

In [5], because of  $m = 1$ ,  $G(\mu)$  is solvable, and thus the discussion depends heavily on the explicit form of  $G(\mu)$ ; whereas, here,  $G(\mu)$  can only be represented implicitly. To overcome this obstacle, we have to seek a different approach.

Although the integrand in (2.3) has a singular point at  $\eta = \mu$ ,  $F(\mu) - F(\eta) \geq \delta(\mu - \eta)$  for some  $\delta > 0$  and  $\eta$  near  $\mu$ ; thus  $G(\mu)$  is continuous for  $0 \leq \mu < 1$ .

Moreover, by rewriting  $G(\mu)$ , one can easily verify that  $G(\mu)$  is actually twice differentiable on  $(0,1)$ , which will be shown later. Knowing some properties of  $G'(\mu)$  on  $(0,1)$  will help us investigate the behavior of  $G(\mu)$ . The motivation for this idea comes from a method used in [1] and [7], where the authors studied the stabilization of solutions of a problem similar to ours, but  $m \geq 1$  with nonlinear term being  $u(1-u)(u-a)$ .

For this purpose, we first present two lemmas.

**LEMMA 3.1.**  *$G(\mu)$  is continuously differentiable on  $(0,1)$ , and there are  $\mu_1$  and  $\mu_2$  in  $(0,1)$  with  $\mu_1 < \mu_2$  such that  $G'(\mu) > 0$  on  $(0, \mu_1)$  and  $G'(\mu) < 0$  on  $(\mu_2, 1)$ .*

**PROOF:** To show this, we use the variant of  $G(\mu)$

$$(3.6) \quad G(\mu) = \sqrt{2m} \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu) - F(\tau\mu)}} d\tau.$$

Formally differentiating  $G(\mu)$  yields

$$(3.7) \quad G'(\mu) = \frac{m}{\mu} G(\mu) - \sqrt{\frac{m}{2}} \mu^{m-1} \int_0^1 \frac{\mu F'(\mu) - \tau \mu F'(\tau \mu)}{(F(\mu) - F(\tau \mu))^{3/2}} \tau^{m-1} d\tau.$$

For  $\mu \in (0, 1)$ , we have  $F'(\mu) = \mu^{m-1} f(\mu)$ . Applying Cauchy mean value theorem, we can see that

$$\frac{\mu F'(\mu) - \tau \mu F'(\tau \mu)}{F(\mu) - F(\tau \mu)} = \frac{m \xi^{m-1} f(\xi) + \xi^m f'(\xi)}{\xi^{m-1} f(\xi)} = m + \frac{\alpha \xi}{1 - \xi},$$

where  $\xi$  is between  $\tau \mu$  and  $\mu$ . Since  $\frac{\xi}{1-\xi} < \frac{\mu}{1-\mu}$ , the integral in (3.7) is convergent, and consequently,  $G'(\mu) \in C(0, 1)$ .

Then we rewrite  $G'(\mu)$  as follows

$$\begin{aligned} G'(\mu) &= \sqrt{\frac{m}{2}} \frac{1}{\mu} \int_0^\mu \frac{2m(F(\mu) - F(\eta))}{(F(\mu) - F(\eta))^{3/2}} \eta^{m-1} d\eta \\ &\quad - \sqrt{\frac{m}{2}} \frac{1}{\mu} \int_0^\mu \frac{\mu F'(\mu) - \eta F'(\eta)}{(F(\mu) - F(\eta))^{3/2}} \eta^{m-1} d\eta \\ &= \sqrt{\frac{m}{2}} \frac{1}{\mu} \int_0^\mu [(2mF(\mu) - \mu^m f(\mu)) - (2mF(\eta) - \eta^m f(\eta))] \\ &\quad \cdot [F(\mu) - F(\eta)]^{-3/2} \eta^{m-1} d\eta. \end{aligned}$$

Set  $H(v) = 2mF(v) - v^m f(v)$  to have

$$(3.8) \quad G'(\mu) = \sqrt{\frac{m}{2}} \frac{1}{\mu} \int_0^\mu \frac{H(\mu) - H(\eta)}{(F(\mu) - F(\eta))^{3/2}} \eta^{m-1} d\eta;$$

then we turn our attention to  $H(\mu)$ .

First, we claim that there is a  $\mu_1 \in (0, 1)$  so that  $H'(\mu) > 0$  on  $(0, \mu_1)$  and

$H'(\mu) < 0$  on  $(\mu_1, 1)$ . Through a straightforward computation

$$\begin{aligned}
 H'(\mu) &= 2mF'(\mu) - m\mu^{m-1}f(\mu) - \mu^m f'(\mu) \\
 &= m\mu^{m-1}f(\mu) - \mu^m f'(\mu) \\
 &= \mu^{m-1} \left[ \frac{m}{(1-\mu)^\alpha} - \frac{\alpha\mu}{(1-\mu)^{\alpha+1}} \right] \\
 &= \mu^{m-1} \left[ \frac{m - (m+\alpha)\mu}{(1-\mu)^{\alpha+1}} \right].
 \end{aligned}$$

Picking  $\mu_1 = \frac{m}{m+\alpha}$  makes our assertion true.

Next, when checking  $H(\mu)$  at two endpoints, we find that  $H(0) = 0$ ; whereas near 1,  $f(\mu) \sim (1-\mu)^{1-\alpha}$  ( $\alpha \neq 1$ ) or  $-\ln(1-\mu)$  ( $\alpha = 1$ ), hence  $H(\mu) \rightarrow -\infty$  as  $\mu \rightarrow 1^-$ . It follows that there is a  $\mu_2 > \mu_1$  such that  $H(\mu) > 0$  on  $(0, \mu_2)$  and  $H(\mu) < 0$  on  $(\mu_2, 1)$ . Combining two properties of  $H(\mu)$  yields the desired result for  $G'(\mu)$ .

Lemma 3.1 suggests that  $G'(\mu)$  has at least one zero one  $(0,1)$ . If we can show that on  $(0,1)$ ,  $G'(\mu)$  has at most one zero, then it follows that there exists a  $\bar{\mu}$  in  $(0,1)$  such that  $G'(\mu)$  is positive on  $(0, \bar{\mu})$  and negative on  $(\bar{\mu}, 1)$ . To this end, we made another statement:

**LEMMA 3.2.**  *$G(\mu)$  is twice differentiable on  $(0,1)$ , and there is a constant  $\alpha^* > 0$ , when  $\alpha \leq \alpha^*$ , for any zero  $\mu_0$  of  $G'(\mu)$ , we have that  $G''(\mu_0) < 0$ .*

**PROOF:**  $G(\mu)$  can be written in the form

$$\frac{G(\mu)}{\sqrt{2m}} = \int_0^\mu \frac{\eta^{m-1} f(\eta)}{f(\eta) \sqrt{F(\mu) - F(\eta)}} d\eta.$$

Upon integration by parts, we obtain

$$\begin{aligned}
 (3.9) \quad \frac{G(\mu)}{\sqrt{2m}} &= \frac{2\sqrt{F(\mu)}}{f(0)} - 2 \int_0^\mu \frac{f'(\eta)}{f^2(\eta)} \sqrt{F(\mu) - F(\eta)} \, d\eta \\
 &= 2\sqrt{F(\mu)} - 2\alpha \int_0^\mu (1-\eta)^{\alpha-1} \sqrt{F(\mu) - F(\eta)} \, d\eta.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \frac{G'(\mu)}{\sqrt{2m}} &= \frac{\mu^{m-1}f(\mu)}{\sqrt{F(\mu)}} - \alpha\mu^{m-1} \int_0^\mu \frac{(1-\eta)^{\alpha-1}}{\sqrt{F(\mu) - F(\eta)}} \, d\eta \\
 &= \mu^{m-1}f(\mu) \left[ \frac{1}{\sqrt{F(\mu)}} - \alpha \int_0^\mu \frac{(1-\eta)^{\alpha-1}}{\sqrt{F(\mu) - F(\eta)}} \, d\eta \right] \\
 &= \mu^{m-1}f(\mu)K(\mu),
 \end{aligned}$$

where  $K(\mu) = \frac{1}{\sqrt{F(\mu)}} - \alpha \int_0^\mu \frac{(1-\eta)^{\alpha-1}}{\sqrt{F(\mu) - F(\eta)}} \, d\eta$ .

Rewrite  $K(\mu)$  as

$$\begin{aligned}
 K(\mu) &= \frac{1}{\sqrt{F(\mu)}} - \alpha \int_0^\mu \frac{\eta^{1-m}(1-\eta)^{\alpha-1}\eta^{m-1}f(\eta)}{f(\eta)\sqrt{F(\mu) - F(\eta)}} \, d\eta \\
 &= \frac{1}{\sqrt{F(\mu)}} - 2\alpha \int_0^\mu \left[ \frac{d}{d\eta} (\eta^{1-m}(1-\eta)^{2\alpha-1}) \right] \sqrt{F(\mu) - F(\eta)} \, d\eta \\
 &= \frac{1}{\sqrt{F(\mu)}} - 2\alpha \int_0^\mu [(1-m) - (2\alpha-m)\eta] \\
 &\quad \cdot \eta^{-m}(1-\eta)^{2\alpha-2} \sqrt{F(\mu) - F(\eta)} \, d\eta,
 \end{aligned}$$

hence  $K(\mu)$  is differentiable, and it follows that  $G''(\mu)$  exists on  $(0,1)$ .

Differentiate  $K(\mu)$  to find

$$K'(\mu) = -\frac{\mu^{m-1}f(\mu)}{2(F(\mu))^{3/2}} - \alpha\mu^{m-1}f(\mu) \int_0^\mu \frac{[(1-m) - (2\alpha-m)\eta^{-m}(1-\eta)^{2\alpha-2}]}{\sqrt{F(\mu) - F(\eta)}} \, d\eta.$$



If  $\alpha \leq \frac{m}{2}$ ,  $(1-m) - (2\alpha-m)\eta > 0$ ; whereas if  $\alpha > \frac{m}{2}$ , for  $\eta \leq \mu_2$  ( $\mu_2$  is the number in Lemma 3.1),  $(1-m) - (2\alpha-m)\eta \geq (1-m) - (2\alpha-m)\mu_2$ .

Let  $(1-m) - (2\alpha-m)\mu_2 \geq 0$ , and it follows that  $\alpha \leq \frac{m}{2} + \frac{(1-m)}{2\mu_2}$ . Set  $\alpha^* = \frac{m}{2} + \frac{(1-m)}{2\mu_2}$ ; then, when  $\alpha \leq \alpha^*$ , we have that  $K'(\mu) < 0$  for  $\mu \leq \mu_2$ .

At any point  $\mu_0$  where  $G'(\mu) = 0$ , we can see that  $G''(\mu_0) = \sqrt{2m} \mu_0^{m-1} f(\mu_0) K'(\mu_0)$ . The proof is complete.

**REMARK 3.1:** Obviously,  $\alpha^* > \frac{1}{2}$ . If  $m$  is small, then  $\alpha^*$  can be very large. For instance, suppose  $\mu_2 = \frac{2m}{m+\alpha}$ ; by elementary calculation, we find that  $(5m-1)\alpha \leq m^2 + m$ . If  $m \leq \frac{1}{5}$ , say, then there is no bound for  $\alpha^*$ .

Relying upon these two lemmas, we are able to prove the following:

**THEOREM 3.3.** *Let  $\alpha \leq \alpha^*$ . Then*

- (i) *If  $0 < \alpha < 1$ , there are two positive numbers  $\varepsilon_1(\alpha)$  and  $\varepsilon_2(\alpha)$  with  $\varepsilon_1(\alpha) < \varepsilon_2(\alpha)$  such that for (II) there is one and only one positive solution if  $0 < \varepsilon \leq \varepsilon_1(\alpha)$  or  $\varepsilon = \varepsilon_2(\alpha)$ , two solutions if  $\varepsilon_1(\alpha) < \varepsilon < \varepsilon_2(\alpha)$ , and no solution for  $\varepsilon > \varepsilon_2(\alpha)$ .*
- (ii) *If  $\alpha \geq 1$ , there is  $\varepsilon(\alpha) > 0$  so that problem (II) has two positive solutions for  $0 < \varepsilon < \varepsilon(\alpha)$ , one solution if  $\varepsilon = \varepsilon(\alpha)$ , and none for  $\varepsilon > \varepsilon(\alpha)$ .*

**PROOF:** Taking note of the previous lemmas, we only need to check the values of  $G(\mu)$  at  $\mu = 0$  and  $\mu = 1$ .

For all cases, it is clear that  $G(0) = 0$  from (3.9).

If  $0 < \alpha < 1$ , we can estimate

$$G(\mu) = \sqrt{2m} \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu) - F(\tau)}} d\tau \geq \sqrt{2m} \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu)}} d\tau.$$

As  $\mu \rightarrow 1^-$ ,  $F(\mu) = \int_0^\mu s^{m-1}(1-s)^{-\alpha} ds \rightarrow \int_0^1 s^{m-1}(1-s)^{-\alpha} ds = B(m, 1-\alpha)$ .

Hence  $\lim_{\mu \rightarrow 1^-} G(\mu) \geq \sqrt{\frac{2}{m}} B(m, 1-\alpha)^{-1/2} > 0$ .

For  $\alpha \geq 1$ , we observe that

$$\begin{aligned} F(\mu) - F(\eta) &= \mu^m \int_0^1 \tau^{m-1}(1-\mu\tau)^{-\alpha} d\tau - \eta^m \int_0^1 \tau^{m-1}(1-\eta\tau)^{-\alpha} d\tau \\ &\geq (\mu^m - \eta^m) \int_0^1 \tau^{m-1}(1-\mu\tau)^{-\alpha} d\tau. \end{aligned}$$

Thus

$$\begin{aligned} G(\mu) &\leq \sqrt{2m} \left[ \int_0^1 \tau^{m-1}(1-\mu\tau)^{-\alpha} d\tau \right]^{-1/2} \int_0^\mu \frac{\eta^{m-1}}{\sqrt{\mu^m - \eta^m}} d\eta \\ &= \sqrt{\frac{2}{m}} 2\mu^{m/2} \left[ \int_0^1 \tau^{m-1}(1-\mu\tau)^{-\alpha} d\tau \right]^{-1/2}. \end{aligned}$$

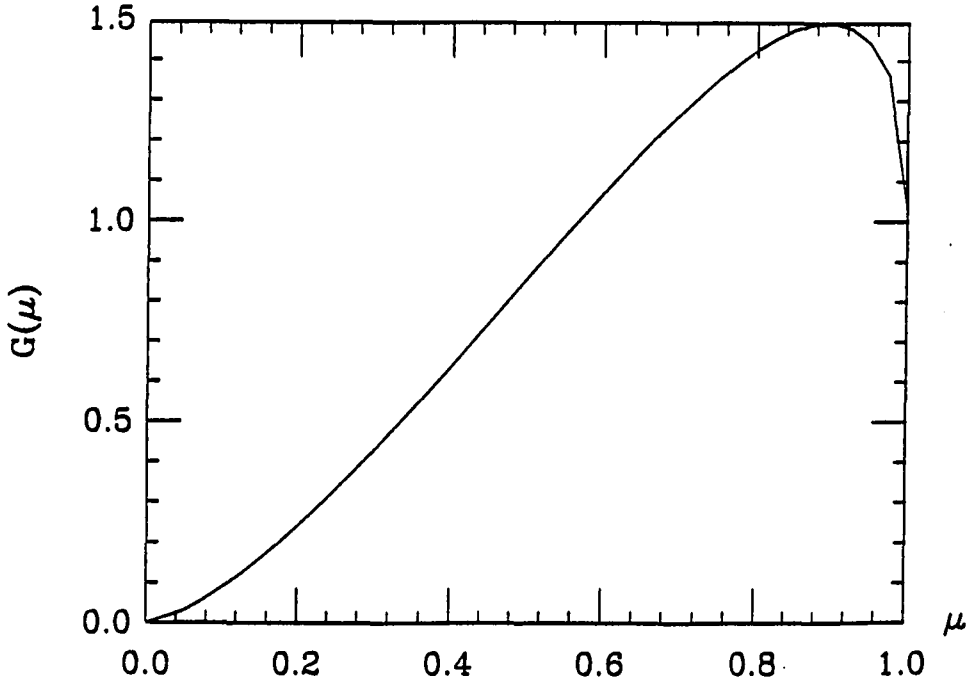


Fig. 1  $G(\mu)$  for  $m=3$ ,  $\alpha=1/2$ .

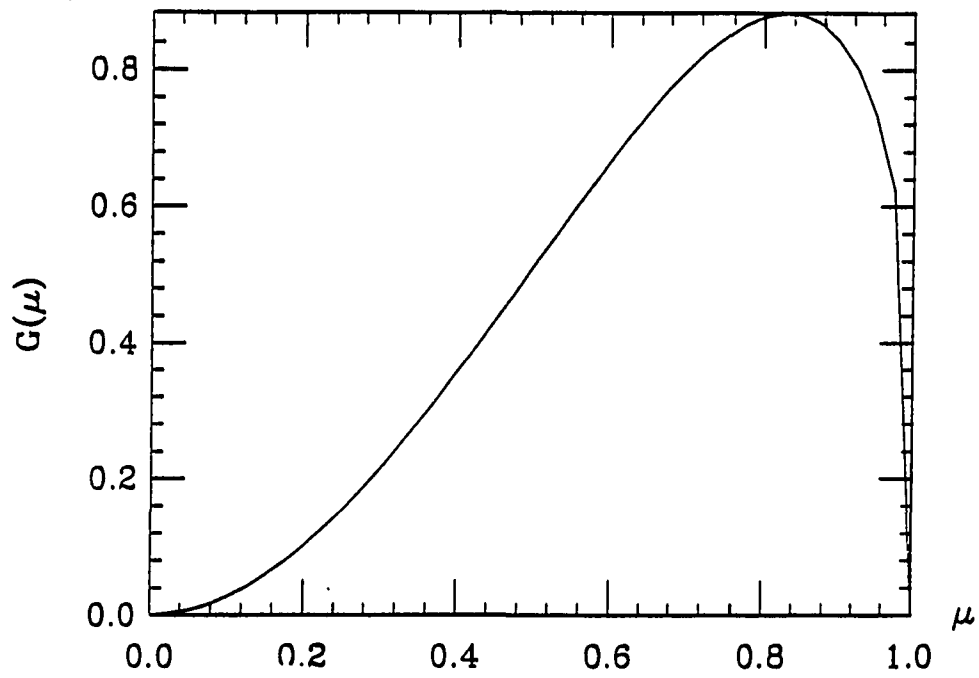


Fig. 2  $G(\mu)$  for  $m = 4, \alpha = 1$ .

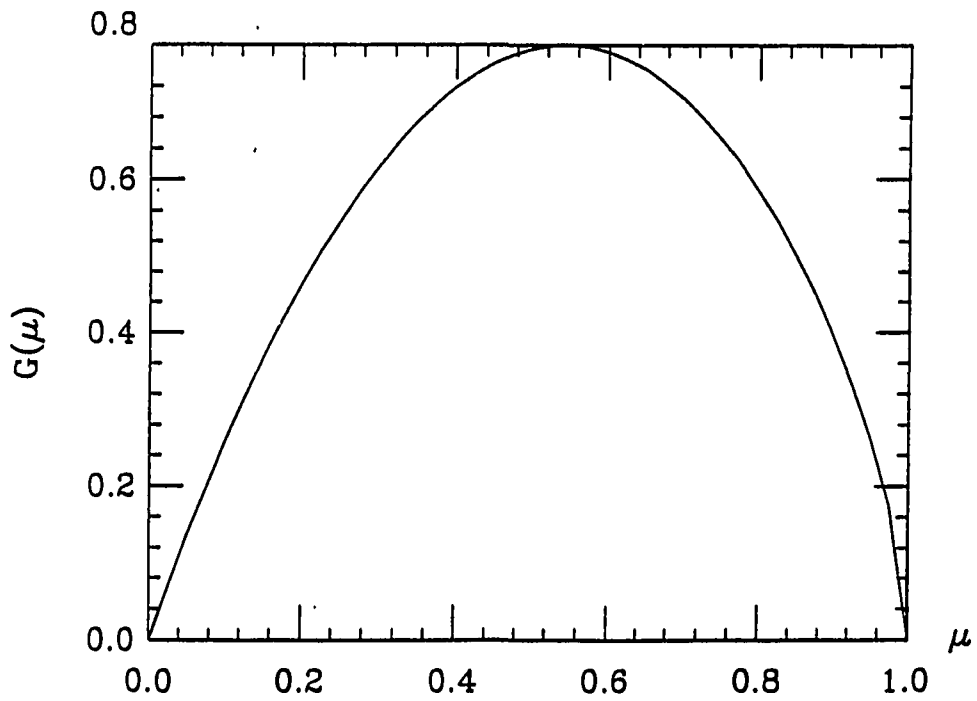


Fig. 3  $G(\mu)$  for  $m = 2, \alpha = 2$ .

Since  $\left[ \int_0^1 \tau^{m-1} (1 - \mu\tau)^{-\alpha} d\tau \right]^{-1/2}$  goes to zero as  $\mu \rightarrow 1^-$ , it follows that  $\lim_{\mu \rightarrow 1^-} G(\mu) = 0$ .

**REMARK 3.2:** Regardless of the restriction on  $\alpha$ , our result is consistent with that in [5]. Thus, we may conjecture that for any  $\alpha > 0$ , Theorem 3.3 holds. Meanwhile, numerical evidence (see Figs. 1–3) supports the conjecture for the porous medium problem.

Now we recount some important properties of  $v(x)$ .

**THEOREM 3.4.**

- (i) On any interval  $I = (\varepsilon_a, \varepsilon_b)$  where  $\mu(\varepsilon) = v(\frac{1}{2}, \varepsilon)$  satisfying (3.4) is continuous in  $\varepsilon$ ,  $v(x, \varepsilon)$  is a continuous function of  $\varepsilon$  in the uniform norm.
- (ii) On any interval  $I$  where  $\mu(\varepsilon)$  is strictly increasing,  $v(x, \varepsilon)$  is strictly increasing in  $\varepsilon$  for all  $x \in (0, 1)$ .
- (iii) If  $\mu_-(\varepsilon) < \mu_+(\varepsilon)$ , then the corresponding solutions are ordered, i.e.  $v_-(x, \varepsilon) < v_+(x, \varepsilon)$  on  $(0, 1)$ .

The proof is virtually identical to that of Theorem 2.1A in [5] and, therefore, is omitted.

Finally, we study the existence of nonclassical solutions of (II), which satisfy

$$(3.10) \quad \int_{\Omega} (\varphi'' v^m + \varepsilon \varphi f(v)) dx = 0 \quad \text{for } \varphi \in C_0^2(\overline{\Omega}).$$

By taking a suitable sequence of  $\varphi$ 's, we see that (3.10) holds if and only if

$$(3.11) \quad v^m = \varepsilon \int_0^1 G(x, y) f(v(y)) dy,$$

where

$$G(x, y) = \begin{cases} x(1-y) & 0 \leq x \leq y \leq 1, \\ y(1-x) & 0 \leq y \leq x \leq 1, \end{cases}$$

is the Green's function for  $-\frac{d^2}{dy^2}$  with Dirichlet boundary condition. In (3.11),  $(1-y)f(v(y))$  and  $yf(v(y))$  are in  $L^1_{\text{Loc}}(\Omega)$ ; hence,  $v$  is absolutely continuous. Whenever  $v(x) < 1$ ,  $v$  satisfies the equation

$$(v^m)'' + \varepsilon f(v) = 0$$

in a classical sense.

For  $\alpha \geq 1$ ,  $F(\mu) = \mu^m \int_0^1 \tau^{m-1} (1 - \mu\tau)^{-\alpha} d\tau \rightarrow \infty$  as  $\mu \rightarrow 1^-$ . If a nonclassical solution exists, then  $v(x) = 1$  at some point  $\xi$ , from (3.1), we see that this is impossible.

For  $0 < \alpha < 1$ , since both  $v(x)$  and  $v(1-x)$  satisfy (3.1), it follows that  $v$  is symmetric with respect to  $x = \frac{1}{2}$ . If  $v(\xi) = 1$ , at some  $\xi$  in  $(0,1)$ , then  $\xi$  should be  $\frac{1}{2}$ . Therefore,  $v(x)$  must be a solution of the integral equation

$$\sqrt{\frac{m}{2}} \int_0^{v(x)} \frac{\eta^{m-1}}{\sqrt{F(1) - F(\eta)}} d\eta = \sqrt{\varepsilon} x,$$

with  $v(\frac{1}{2}) = 1$  and  $\varepsilon$  satisfying

$$\sqrt{2m} \int_0^1 \frac{\eta^{m-1}}{\sqrt{F(1) - F(\eta)}} d\eta = \sqrt{\varepsilon}.$$

Consequently,  $\varepsilon = \varepsilon_1(\alpha)$ , which is the critical number in Theorem 3.3.

To sum up, we conclude:

**THEOREM 3.5.** *If  $\alpha \geq 1$  (II) has no nonclassical solution. For  $0 < \alpha < 1$ , there is exactly one solution  $v_*$  when  $\varepsilon = \varepsilon_1(\alpha)$ .*

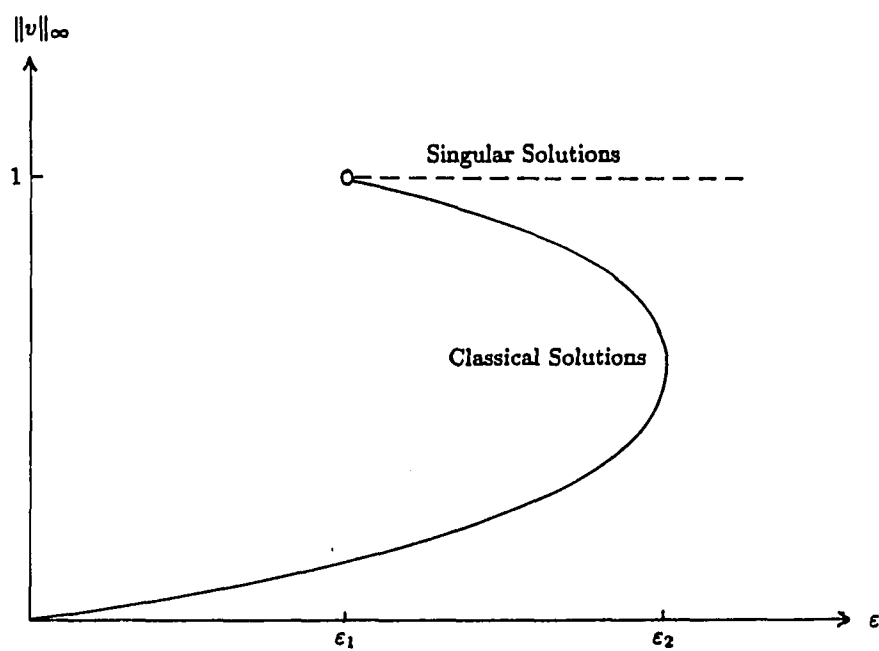


Fig. 4 Bifurcation diagram for  $0 < m < 1$ ,  $0 < \alpha < 1$ .

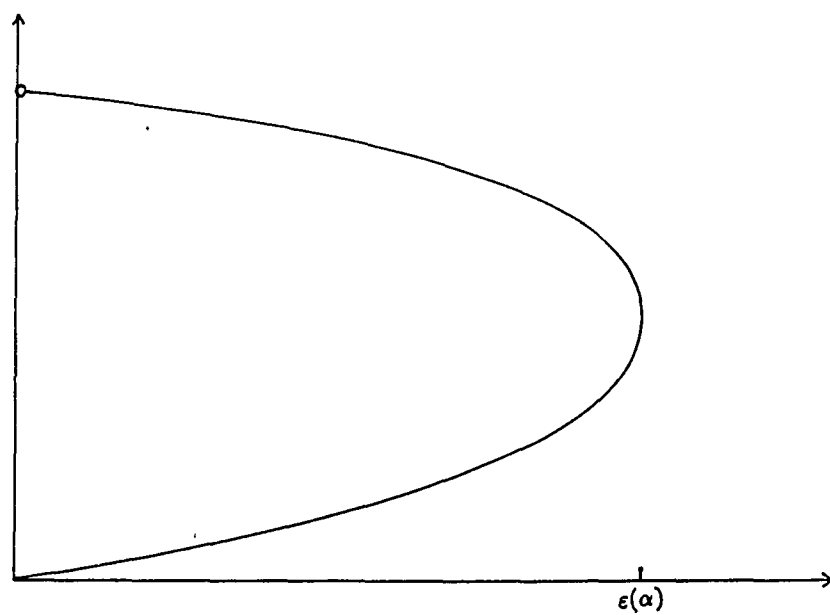


Fig. 5 Bifurcation diagram for  $0 < m < 1$ ,  $1 \leq \alpha$ .

**REMARK 3.3:** For singular stationary solutions of the following problem:

$$\begin{aligned}
 (I^*) \quad & u_t = (u^m)_{xx} + \varepsilon(1-u)^{-\alpha}\chi(u < 1) \quad 0 < x < 1, \quad t > 0, \\
 & u(0, t) = u(1, t) = 0 \quad t > 0, \\
 & u(x, 0) = u_0(x) \quad 0 \leq x \leq 1,
 \end{aligned}$$

here  $\chi(A)$  is the characteristic function of the set  $A$ , we must have

$$(3.12) \quad v^m = \varepsilon \int_0^1 G(x, y) f(v(y)) \chi(v < 1) dy.$$

Since  $v$  is classical wherever  $v < 1$  and  $v$  is symmetric with respect to  $x = \frac{1}{2}$ , the same argument as above can apply, and it follows that if  $0 < \alpha < 1$  and  $\varepsilon \geq \varepsilon(\alpha)$ , then there is exactly one singular stationary solution of  $(I^*)$ .

Bifurcation diagrams of the stationary solutions are given in Figs. 4 and 5.

#### 4. Quenching and Global Existence

In order to get statements about quenching or stability of solutions, we first establish that if  $u(x, t)$  is a solution of (I) such that

$$\lim_{t \rightarrow \infty} u(x, t) = v(x),$$

then  $v(x)$  is a solution of (II). In this direction, we begin with a certain integral equation for the solution of (I).

**LEMMA 4.1.** *Let  $u(x, t)$  be a solution of (I). Then*

$$\begin{aligned}
 (4.1) \quad & \int_0^t \left[ u^m(x, \tau) - \varepsilon \int_0^1 G(x, y) f(u(y, \tau)) dy \right] d\tau \\
 & = \int_0^1 G(x, y) [u(y, t) - u_0(y)] dy,
 \end{aligned}$$

where  $G(x, y)$  is the Green's function for  $-\frac{d^2}{dy^2}$ .

PROOF: From the definition for  $u$ ,

$$\begin{aligned} & \int_0^1 u(y, T) \varphi(y, T) dy - \int_0^T \int_0^1 (u \varphi_t + u^m \varphi_{yy}) dy dt \\ &= \int_0^1 u_0 \varphi(y, 0) dy + \varepsilon \int_0^T \int_0^1 f(u) \varphi dy dt. \end{aligned}$$

Consider  $\{\varphi_n\}_{n=1}^\infty$ , a sequence with  $\varphi_n(x, y, t) = \psi_n(x, y) \theta(t)$ , where  $\psi_n(\cdot, y) \in C_0^\infty(\bar{\Omega})$ ,  $\psi_n(x, y) \rightarrow G(x, y)$ ,  $\psi_{nyy} \rightarrow G_{yy}$ , and  $\theta(t) \in C^\infty([0, T])$  with  $\theta(T) = 0$ .

We then have

$$\begin{aligned} & - \int_0^T \int_0^1 u \psi_n \theta'(t) dy dt - \int_0^T \int_0^1 u^m \psi_{nyy} \theta(t) dy dt \\ &= \int_0^1 u_0 \psi_n \theta(0) dy + \varepsilon \int_0^T \int_0^1 f(y) \psi_n \theta(t) dy dt. \end{aligned}$$

Integration by parts produces

$$\int_0^T \left[ \int_0^1 (u - u_0) \psi_n dy + \int_0^t \int_0^1 u^m \psi_{nyy} dy d\tau + \varepsilon \int_0^t \int_0^1 f(y) \psi_n dy d\tau \right] \theta'(t) dt = 0.$$

Since  $\theta'(t)$  can be chosen arbitrarily, it follows that

$$\int_0^1 \psi_n (u - u_0) dy + \int_0^t \int_0^1 \psi_{nyy} u^m dy d\tau + \varepsilon \int_0^t \int_0^1 \psi_n f(u) dy d\tau = 0.$$

Letting  $n \rightarrow \infty$ , we obtain the form (4.1).

Using this integral equation for  $u$ , we can show:



LEMMA 4.2. Suppose that  $u(x, t)$  is monotone in  $t$  and  $\lim_{t \rightarrow \infty} u(x, t) = v(x)$  exists. Then  $v(x)$  is a solution of (II).

PROOF: Since  $u(x, t)$  is uniformly bounded and monotone in  $t$ , for fixed  $x \in [0, 1]$ ,

$$\lim_{t \rightarrow \infty} \int_0^1 G(x, y) u(y, t) dy = \int_0^1 G(x, y) v(y) dy;$$

and then

$$\frac{\partial}{\partial t} \int_0^1 G(x, y) u(y, t) dy \rightarrow 0$$

for some sequence  $t_n \rightarrow \infty$ . The equation (4.1) implies that

$$v^m(x) = \varepsilon \int_0^1 G(x, y) f(v(y)) dy,$$

which is exactly the same as (3.9); and hence  $v(x)$  is a solution of (II).

With Lemma 4.2 in hand, we are ready to establish quenching and nonquenching results for problem (I).

THEOREM 4.3. Let  $0 < \alpha < 1$ .

- (i) For  $0 < \varepsilon < \varepsilon_1$ , if  $0 \leq u_0 < v_s(x, \varepsilon_1)$ , then quenching does not occur even in infinite time and  $\lim_{t \rightarrow \infty} u(x, t) = v(x, \varepsilon)$ .
- (ii) For  $\varepsilon = \varepsilon_1$ , we have the following:
  - (a) If  $0 \leq u_0 \leq v(x, \varepsilon_1)$ , then  $u(x, t)$  is global and  $\lim_{t \rightarrow \infty} u(x, t) = v(x, \varepsilon)$ .
  - (b) If  $v(x, \varepsilon_1) < u_0 < v_s(x, \varepsilon_1)$ , then quenching cannot happen and  $\lim_{t \rightarrow \infty} u(x, t) = v(x, \varepsilon_1)$ .
- (iii) For  $\varepsilon_1 < \varepsilon < \varepsilon_2$ , the following hold:
  - (a) If  $0 \leq u_0 < v_+(x, \varepsilon)$ , then  $u$  exists globally and  $\lim_{t \rightarrow \infty} u(x, t) = v_-(x, \varepsilon)$ .
  - (b) If  $v_+(x, \varepsilon) < u_0$ , then  $u$  quenches in finite time.

(iv) For  $\varepsilon = \varepsilon_2$ , we have the following:

(a) If  $0 \leq u_0 \leq v(x, \varepsilon_2)$ , then  $u$  is a global solution and  $\lim_{t \rightarrow \infty} u(x, t) = v(x, \varepsilon_2)$ .

(b) If  $v(x, \varepsilon_2) < u_0$ , then  $u$  only exists locally.

(v) For  $\varepsilon > \varepsilon_2$ , every solution of (I) with  $0 \leq u_0 < 1$  quenches in finite time.

PROOF: We follow closely the proof for Theorem 3.2A of [5] given in the case  $m = 1$  by Levine. Without any confusion, sometimes we shall write the solution of (I) with an initial value  $u_0$  as  $u(x, t; \varepsilon, u_0)$ .

(i) We choose  $v_0 \equiv 0$  and  $w_0 = v_+(x, \delta)$  where  $\varepsilon_1 < \delta$  is so close to  $\varepsilon_1$  that  $u_0 \leq w_0$  on  $(0, 1)$ . Then, by comparison,

$$u(x, t; \varepsilon, v_0) \leq u(x, t; \varepsilon, u_0) \leq u(x, t; \varepsilon, w_0).$$

On the other hand, by recalling Theorem 2.3, we can see that  $u(x, t; \varepsilon, v_0)$  is monotonically increasing; whereas  $(w_0^m)'' + \varepsilon f(w_0) < (w_0^m)'' + \delta f(w_0) = 0$  implies that  $u(x, t; \varepsilon, w_0)$  is monotonically decreasing. Hence, both  $\lim_{t \rightarrow \infty} u(x, t; \varepsilon, v_0)$  and  $\lim_{t \rightarrow \infty} u(x, t; \varepsilon, w_0)$  exist, and they are equal to  $v(x, \varepsilon)$ , the unique stationary solution; consequently, so is  $\lim_{t \rightarrow \infty} u(x, t; \varepsilon, u_0)$ .

(ii) (a) Because  $u(x, t; \varepsilon_1, 0) \leq u(x, t; \varepsilon_1, u_0) \leq v(x, \varepsilon_1)$ , the conclusion follows.

(b) Proof for (b) is similar to that for (i) and hence is omitted.

(iii) (a) By the same reason as for (b) of (ii), the proof is left out.

(b) Set  $v_0 = v_+(x, \delta)$  with  $\delta < \varepsilon$  such that  $v_+(x, \varepsilon) < v_0 < u_0$  to find that  $(v_0^m)'' + \varepsilon f(v_0) > (v_0^m)'' + \delta f(v_0) = 0$ . Then,  $u(x, t; \varepsilon, v_0)$  is monotonically increasing and  $u(x, t; \varepsilon, v_0) \leq u(x, t; \varepsilon, u_0)$ . Suppose that  $u(x, t; \varepsilon, u_0)$  existed for all  $t > 0$ ; then  $u(x, t; \varepsilon, v_0)$  would exist, too, and it would follow that  $\lim_{t \rightarrow \infty} u(x, t; \varepsilon, v_0)$  exists, and should be greater than  $v_+(x, \varepsilon)$ . Because there is no nonclassical solution for  $\varepsilon > \varepsilon_1$ , a contradiction results.

(iv) (a) Letting  $v_0 \equiv 0$  makes the assertion hold.

The proofs for (b) of (iv) and (v) are the same as that for (b) of (iii).

For the case  $\alpha \geq 1$ , we can argue in a similar manner. Thus, we only list the results as follows:

**THEOREM 4.4.** *Let  $1 \leq \alpha \leq \alpha^*$ .*

(i) *For  $0 < \varepsilon < \varepsilon(\alpha)$ , we have the following:*

(a) *If  $0 \leq u_0 < v_+(x, \varepsilon)$ , then  $u$  is global and  $\lim_{t \rightarrow \infty} u(x, t) = v_-(x, \varepsilon)$ .*

(b) *If  $v_+ < u_0$ , then  $u$  is not global.*

(ii) *For  $\varepsilon = \varepsilon(\alpha)$ , the following statements hold:*

(a) *If  $0 \leq u_0 \leq v(x, \varepsilon(\alpha))$ , then  $u$  exists for all  $t > 0$ , and  $\lim_{t \rightarrow \infty} u(x, t) = v(x, \varepsilon(\alpha))$ .*

(b) *If  $v(x, \varepsilon(\alpha)) < u_0$ , then  $u$  quenches in finite time.*

(iii) *For  $\varepsilon > \varepsilon(\alpha)$ , every solution quenches in finite time.*

Finally, without being involved in any discussion about a stationary solution of problem (I), we can still obtain some quenching results. The main idea, as shown in [2] and [5], is to use the first eigenvalue of the Dirichlet boundary value problem.

**THEOREM 4.5.** *Let  $\lambda_1$  be the first eigenvalue of  $-\frac{d^2}{dx^2}$  and let  $\varphi(x)$  be the corresponding eigenfunction. Suppose that one of the following hypotheses holds:*

$$(H1) \int_0^1 u_0 \varphi(x) dx > r_0 = \max_{-\infty < r < 1} r, \text{ where } \varepsilon f(r) - \lambda_1 r^m = 0,$$

or

$$(H2) \varepsilon > \varepsilon_0 = \max_{0 < x < 1} \lambda_1 x^m / f(x).$$

*Then, the solution  $u(x, t)$  of (I) quenches in finite time.*

**PROOF:** Assume the contrary that  $u(x, t)$  exists for all  $t > 0$ . By the definition of

$u$ , put  $\varphi(x, t) \equiv \varphi(x)$  to get

$$\int_0^1 u(x, t) \varphi(x) dx + \lambda_1 \int_0^t \int_0^1 u^m(x, \tau) \varphi(x) dx d\tau = \int_0^1 u_0(x) \varphi(x) dx + \varepsilon \int_0^t \int_0^1 f(u) \varphi(x) dx d\tau.$$

Differentiate the above equation with respect to  $t$

$$\frac{d}{dt} \int_0^1 u(x, t) \varphi(x) dx + \lambda_1 \int_0^1 u^m(x, t) \varphi(x) dx = \varepsilon \int_0^1 f(u) \varphi(x) dx.$$

Setting  $h(t) = \int_0^1 u(x, t) \varphi(x) dx$  and using the concavity of  $u^m$  and  $f(u)$ , we have

$$h'(t) \geq -\lambda_1 h^m + \varepsilon f(h).$$

Whenever H1 or H2 holds,  $h'(t) \geq h'(0) > 0$ ; and it follows that

$$1 > h(t) \geq h(0) + h'(0)t,$$

a contradiction.

For the porous medium problem, note that  $x^m < x$  for  $m > 1$  and  $0 < x < 1$ .

Similar results can be obtained.

**THEOREM 4.6.** *Suppose that one of the following conditions is satisfied:*

$$(C1) \int_0^1 u_0 \varphi(x) dx > r_1 = \max_{-\infty < r < 1} r, \text{ with } \varepsilon f(r) - \lambda_1 r = 0,$$

or

$$(C2) \varepsilon > \varepsilon_1 = \max_{0 < x < 1} \lambda_1 x / f(x).$$

*Then, the solution of the porous medium problem quenches in finite time.*

**REMARK:** The results in the last two theorems are applicable to the  $n$  dimensional space ( $n > 1$ ) case, with a general convex function  $f(u)$ .

## 5. References

- [1] D. Aronson, M. G. Crandall, and L. A. Peletier. *Stablization of solutions of a degenerate nonlinear diffusion problem*. Nonlinear Anal. TMA 6 (1982), 1001–1022.
- [2] K. Deng. *The quenching of solutions of semilinear parabolic equations*. J. HUST. 7 (1985), 1–6.
- [3] K. Deng and H. A. Levine. *On the blowup of  $u_t$  at quenching*. Proc. AMS 106 (1989), 1049–1056.
- [4] J. Filo. *On solutions of a perturbed fast diffusion equation*. Aplikace Matematiky 32 (1987), 364–380.
- [5] H. A. Levine. *Quenching, nonquenching, and beyond quenching for solutions of some parabolic equations*. Ann. Mat. Pura Appl. (IV) 155 (1989), 243–260.
- [6] P. E. Sacks. *Continuity of solutions of a singular parabolic equation*. Nonlinear Anal. TMA 7 (1983), 387–409.
- [7] J. Smoller and J. Wasserman. *Global bifurcations steady-state solutions*. J. Diff. Eqns. 39 (1981), 269–290.

### PART III. BEHAVIOR OF SOLUTIONS OF BURGERS' EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

#### Abstract

In this paper, we discuss the long-time behavior of positive solutions of Burgers' equation  $u_t = u_{xx} + \varepsilon u u_x$ ,  $0 < x < 1$ ,  $t > 0$ ,  $\varepsilon > 0$  with nonlocal boundary conditions: (A)  $u(0, t) = 0$ ,  $u_x(1, t) = au^p(1, t) \left( \int_0^1 u(x, t) dx \right)^q - \frac{1}{2} \varepsilon u^2(1, t)$ , or (B)  $-u_x(0, t) = au^p(0, t) \left( \int_0^1 u(x, t) dx \right)^q + \frac{1}{2} \varepsilon u^2(0, t)$ ,  $u(1, t) = 0$ , where  $0 < p < \infty$ ,  $0 < q < \infty$ . Criteria for stability are given. Blow up in finite time for some solutions is shown. General results are discussed.

#### 1. Introduction

In this paper, we consider two initial nonlocal boundary-value problems:

$$\begin{aligned}
 (A) \quad & u_t = u_{xx} + (f(u))_x & 0 < x < 1, \quad t > 0, \\
 & u(0, t) = 0 & t > 0, \\
 & u_x(1, t) = g(u(1, t), \bar{u}) & t > 0, \\
 & u(x, 0) = u_0(x) & 0 \leq x \leq 1
 \end{aligned}$$

and

$$\begin{aligned}
 (B) \quad & u_t = u_{xx} + (f(u))_x & 0 < x < 1, \quad t > 0, \\
 & -u_x(0, t) = g(u(0, t), \bar{u}) & t > 0, \\
 & u(1, t) = 0 & t > 0, \\
 & u(x, 0) = u_0(x) & 0 \leq x \leq 1,
 \end{aligned}$$

where  $u_0(x)$  are nonnegative prescribed functions and  $\bar{u} = \int_0^1 u(x, t) dx$ ,  $f$  and  $g$  are continuously differentiable functions satisfying  $f(0) = 0$  and  $g(0, v) = 0$ . In particular, when  $f(u) = \frac{1}{2}\varepsilon u^2$  ( $\varepsilon > 0$ ),  $g(u, v) = au^p v^q$  with  $p > 0$ ,  $q > 0$  or  $g(u, v) = au^p v^q - f(u)$  in (A) and  $au^p v^q + f(u)$  in (B), the above problems reduce to Burgers' equation with nonlocal boundary conditions. Of more interest are the last two cases, which—because boundary conditions represent a forcing of the flux—arise from the physical motivation.

In a recent paper [8], Levine presented results concerning the asymptotic behavior of positive solutions of Burgers' equation  $u_t = u_{xx} + \varepsilon u u_x$ ,  $0 < x < 1$ ,  $t > 0$ , subject to either boundary condition (A)  $u(0, t) = 0$ ,  $u_x(1, t) = au^p(1, t)$ ,  $t > 0$  or (B)  $-u_x(0, t) = au^p(0, t)$ ,  $u(1, t) = 0$ ,  $t > 0$  with  $a > 0$  and  $0 < p < \infty$ . He analyzed stability and instability of the stationary solutions and showed that some solutions blow up in finite time. Recently, Anderson [1] and Park [13] generalized these results in two different directions. In [1], the porous medium equation  $u_t = (u^m)_{xx} + \frac{\varepsilon}{n}(u^n)_x$  is discussed; while in [13] the case  $p < 0$  is considered.

Matano [12] surveyed the general problems in a strongly order-preserving framework. The resulting theory has wide application but does not include the case in which nonlocal boundary conditions are involved.

On the other hand, Day ([2],[3]), Friedman [4], and Kawohl [6] obtained results for  $u_t = Lu$  with a nonlocal boundary condition:  $u|_{\partial\Omega} = \int_{\Omega} f(x)u(x, t)dx$ , which for our problem reads  $u(0, t) = u(1, t) = \int_0^1 f(x)u(x, t)dx$ . They established the existence and monotonic decay property of solutions.

Our purpose is to study the existence and monotonicity of solutions and their stable and unstable behavior. In particular, some complete analyses are made for Burgers' equation.

In the second section, we discuss several important properties for corresponding

stationary solutions of (A) and (B) and characterize the set of stationary solutions of Burgers' equation. In §3, we establish comparison theorems for (A) and (B) and discuss the local existence and blow up phenomenon of solutions. Finally, we analyze the stability behavior and give the solution diagrams for Burgers' equation in §4. Because of the nonlocal term in the boundary condition, the discussion becomes complicated, and certain conventional arguments do not apply. Some problems remain unsolved. For instance, in Burgers' equation with  $g(u, v) = au^p v^q \pm \frac{1}{2}\varepsilon u^2$ , our results partly rely on numerical experiments. However, all results for purely local boundary conditions with  $g(u) = au^p$  are covered. Especially, the comparison principle obtained in this paper can be used in more general problems, and it is interesting to note that the solution diagrams for Burgers' equation with  $g(u, v) = au^p v^q \pm \frac{1}{2}\varepsilon u^2$  are totally different from those with  $g(u, v) = au^p v^q$ .

## 2. Stationary Solutions and their Basic Properties

The arguments used herein are extensions of those described in [8]. In this section, we concentrate on the stationary solutions of (A) and (B), which solve these problems, respectively:

$$(A_1) \quad \begin{aligned} w''(x) + (f(w))' &= 0 & 0 < x < 1, \\ w(0) &= 0, & w'(1) = g(w(1), \bar{w}) \end{aligned}$$

and

$$(B_1) \quad \begin{aligned} w''(x) + (f(w))' &= 0 & 0 < x < 1, \\ -w'(0) &= g(w(0), \bar{w}), & w(1) = 0. \end{aligned}$$



Here,  $\bar{w} = \int_0^1 w(x) dx$ .

Without placing a monotonicity condition on  $f$  and  $g$ , we have

**LEMMA 2.1A.** *Nontrivial solutions of  $(A_1)$  are of one sign. Furthermore, every positive solution  $w(x)$  of  $(A_1)$  satisfies  $w'(x) > 0$  on  $[0, 1]$ .*

**PROOF:** The first assertion follows from the maximum principle. As for the second statement, if  $w'(x)$  changed sign on  $(0, 1)$ ,  $w$  would have an interior extremum, which cannot happen unless  $w(x) \equiv \text{constant} = 0$ . Since  $w'(0) > 0$ , it follows that  $w'(x) \geq 0$  on  $[0, 1]$ . Suppose that for some  $x_0 \in (0, 1]$ ,  $w'(x) > 0$  in  $(0, x_0)$  and  $w'(x_0) = 0$ . Then  $w(x_0) > w(x)$  for  $x$  in  $(0, x_0)$ , and the Hopf boundary point lemma yields  $w'(x_0) > 0$ , which is a contradiction.

Using the same argument, we can show

**LEMMA 2.1B.** *Nontrivial solutions of  $(B_1)$  are of one sign. Moreover, every positive solution  $w(x)$  of  $(B_1)$  satisfies  $w'(x) < 0$  on  $[0, 1]$ .*

$(A_1)$  may have more than one solution, but adding a further assumption on  $f$ , we can find that solutions are ordered.

**LEMMA 2.2A.** *If  $f'(u) \geq 0$  for  $u > 0$ ,  $w_1(x)$  and  $w_2(x)$  are two positive solutions of  $(A_1)$  that satisfy  $w_1(1) > w_2(1)$ , then  $w_1(x) > w_2(x)$  on  $(0, 1]$ .*

**PROOF:** Assume that the statement is not true. There is then an  $x_0$  nearest 1 such that  $w_1(x_0) = w_2(x_0)$  and  $w_1(x) > w_2(x)$  for  $x$  in  $(x_0, 1]$ . Hence we can find a least  $x_1$  in  $(x_0, 1]$ , so that  $w'_1(x_1) > w'_2(x_1)$ . From the conservation law

$$w'_1(0) = w'_1(x) + f(w_1(x)) = w'_1(x_1) + f(w_1(x_1))$$

and

$$w'_2(0) = w'_2(x) + f(w_2(x)) = w'_2(x_1) + f(w_2(x_1)),$$

it follows that  $w_1'(0) > w_2'(0)$ .

Because  $w_1(0) = w_2(0) = 0$ , we know that  $w_1(x) > w_2(x)$  near 0. Suppose that  $x_2$  is the first point, such that  $w_1(x) > w_2(x)$  in  $(0, x_2)$  and  $w_1(x_2) = w_2(x_2)$ . From the above identities, we find that  $w_1'(x_2) > w_2'(x_2)$ . For  $x$  in  $(0, x_2)$ , however,

$$\frac{w_1(x) - w_1(x_2)}{x - x_2} < \frac{w_2(x) - w_2(x_2)}{x - x_2},$$

which implies that  $w_1'(x_2) \leq w_2'(x_2)$ —a contradiction.

For problem  $(B_1)$ , we obtain the same conclusion by placing an additional restriction on  $f$ .

**LEMMA 2.2B.** *If  $f$  is twice continuously differentiable,  $f'(u)$  is strictly increasing for  $u > 0$ , and  $w_1(x)$ ,  $w_2(x)$  are two positive solutions of  $(B_1)$  that satisfy  $w_1(0) > w_2(0)$ , then  $w_1(x) > w_2(x)$  on  $[0, 1)$ .*

**PROOF:** Let  $w(x) = w_1(x) - w_2(x)$ , then  $w(x)$  satisfies

$$\begin{aligned} w'' + f'(w_1)w' + (f'(w_1) - f'(w_2))w_2' &= 0 \quad 0 < x < 1, \\ w(0) > 0 \quad \text{and} \quad w(1) &= 0. \end{aligned}$$

Because  $f'(u)$  is strictly increasing and  $w_2' < 0$ ,  $w$  cannot have a negative interior minimum. If  $w$  had a zero at  $x_0$  in  $(0, 1)$ , the maximum principle would yield  $w \equiv 0$  on  $[x_0, 1]$ . Hence,  $w'' + f'(w_1)w' + f''(\xi)w_2'w = 0$  with  $w(x_0) = w'(x_0) = 0$  implies that  $w \equiv 0$  on  $(0, 1]$ , which is impossible.

Although we cannot establish an existence theorem for the solutions of  $(A_1)$  or  $(B_1)$ , by applying further conditions to  $f$  and  $g$ , we are sometimes able to obtain the uniqueness of solutions of  $(A_1)$  and  $(B_1)$ .

**THEOREM 2.3A.** *Let  $f'$  be nonnegative and strictly increasing for  $u > 0$ , and suppose that one of two conditions holds: (i)  $g(u_1, v_1)/u_1 \geq g(u_2, v_2)/u_2$  or (ii)*

$g(u_1, v_1)/v_1 \geq g(u_2, v_2)/v_2$  for  $u_1 > u_2 > 0$  and  $v_1 > v_2 > 0$ . Then, at most, one positive solution of  $(A_1)$  exists.

PROOF: Suppose that there are two solutions— $w_1(x)$  and  $w_2(x)$  of  $(A_1)$ —satisfying  $w_1(1) > w_2(1)$ . By Lemma 2.2A, it follows that  $w_1(x) > w_2(x)$  on  $(0, 1]$  and that  $\bar{w}_1 = \int_0^1 w_1(x)dx > \bar{w}_2 = \int_0^1 w_2(x)dx$ . Moreover, we have  $w_1'(x) > 0$  and  $w_2'(x) > 0$  by Lemma 2.1A. Let  $R(x) = w_2'(x)/w_1'(x)$ ,  $R'(x) = (f'(w_1) - f'(w_2))/R > 0$  on  $(0, 1]$ . Therefore, on  $(0, 1)$

$$\frac{w_2'(x)}{w_1'(x)} < \frac{w_2'(1)}{w_1'(1)}.$$

From the above, we find that

$$w_2(x_0) < \frac{w_2'(1)}{w_1'(1)} w_1(x_0) \quad x_0 \in (0, 1)$$

and that

$$w_2(x) - w_2(x_0) < \frac{w_2'(1)}{w_1'(1)} (w_1(x) - w_1(x_0)) \quad x_0 < x < 1;$$

that is,

$$\frac{w_2'(1)}{w_1'(1)} w_1(x) - w_2(x) > \frac{w_2'(1)}{w_1'(1)} w_1(x_0) - w_2(x_0) > 0.$$

Letting  $x \rightarrow 1^-$ , we have

$$\frac{w_2'(1)}{w_1'(1)} w_1(1) - w_2(1) > 0;$$

hence  $\frac{w_2'(1)}{w_1'(1)} > \frac{w_2(1)}{w_1(1)}.$

If (i) holds,

$$\frac{w_2'(1)}{w_1'(1)} = \frac{g(w_2(1), \bar{w}_2)}{g(w_1(1), \bar{w}_1)} = \frac{g(w_2(1), \bar{w}_2)/w_2(1)}{g(w_1(1), \bar{w}_1)/w_1(1)} \cdot \frac{w_2(1)}{w_1(1)} \leq \frac{w_2(1)}{w_1(1)}.$$

Combining the above two inequalities leads to a contradiction:

$$\frac{w_2(1)}{w_1(1)} < \frac{w'_2(1)}{w'_1(1)} \leq \frac{w_2(1)}{w_1(1)}.$$

On the other hand, from  $\frac{w'_2(1)}{w'_1(1)}w_1(x) - w_2(x) > 0$ , it follows that  $\frac{\bar{w}_2}{\bar{w}_1} < \frac{w'_2(1)}{w'_1(1)}$ . If (ii) were valid,

$$\frac{w'_2(1)}{w'_1(1)} = \frac{g(w_2(1), \bar{w}_2)/\bar{w}_2}{g(w_1(1), \bar{w}_1)/\bar{w}_1} \cdot \frac{\bar{w}_2}{\bar{w}_1} \leq \frac{\bar{w}_2}{\bar{w}_1}$$

which is impossible.

By analogous reasoning, we obtain the following result:

**THEOREM 2.3B.** *Let  $f'$  be strictly increasing for  $u > 0$ , and suppose that  $g(u_1, v_1)/u_1 \leq g(u_2, v_2)/u_2$  or  $g(u_1, v_1)/v_1 \leq g(u_2, v_2)/v_2$  for  $u_1 > u_2 > 0$  and  $v_1 > v_2 > 0$ . Then, at most, one positive solution of  $(B_1)$  exists.*

Noteworthy is the equivalence between solutions of  $(A_1)$  or  $(B_1)$  and those obtained from integral equations.

**THEOREM 2.4A.** *Let  $f' > 0$  for  $u > 0$ . Suppose that  $w(x)$  is a positive solution of  $(A_1) \in C^2(0, 1) \cap C^1[0, 1]$ . Then  $w(x)$  satisfies*

$$F(w; \alpha, \beta) = \int_0^{w(x)} \frac{d\sigma}{g(\alpha, \beta) + f(\alpha) - f(\sigma)} = x$$

for  $0 \leq x \leq 1$ . Here,  $\alpha = w(1)$  and  $\beta = \int_0^1 w(x)dx$ .

Conversely, if  $\alpha > 0$  satisfies  $F(\alpha; \alpha, \beta) = 1$ , positive  $w(x)$  satisfies  $F(w; \alpha, \beta) = x$  with  $\int_0^1 w(x)dx = \beta$ , then  $w(x)$  is a positive solution of  $(A_1)$ .

**PROOF:** The first statement follows by quadrature. We prove the second part. First, we claim that  $h(\sigma) = g(\alpha, \beta) + f(\alpha) - f(\sigma) > 0$  for  $\sigma \in [0, \alpha]$ . Otherwise,

suppose that there existed an  $x_0 \in (0, 1)$  with  $\sigma_0 = w(x_0)$  such that  $h(\sigma_0) = 0$ . Then  $h(\sigma) = h(\sigma) - h(\sigma_0) = f(\sigma_0) - f(\sigma) \approx f'(\sigma_0)(\sigma_0 - \sigma)$ , and we would have

$$x_0 = \int_0^{w(x_0)} \frac{d\sigma}{h(\sigma)} = \int_0^{\sigma_0} \frac{d\sigma}{f(\sigma_0) - f(\sigma)} = +\infty$$

which is a contradiction.

Next we show that  $w(x) < \alpha$  for  $0 \leq x < 1$ . If not, there would be at least a  $x_1 \in [0, 1)$  such that  $w(x_1) \geq \alpha$ . Because  $h(\sigma) > 0$ ,

$$1 = \int_0^\alpha \frac{d\sigma}{h(\sigma)} \leq \int_0^{w(x_1)} \frac{d\sigma}{h(\sigma)} = x_1,$$

which cannot happen. This argument implies that  $w(1) = \alpha$ .

We see that  $F(w; \alpha, \beta)$  is twice continuously differentiable in  $w$  for  $0 \leq w \leq w(1)$ . Thus, by the inverse function theorem,  $w(x)$  is twice differentiable, and we obtain

$$w'(x) + f(w(x)) = g(w(1), \bar{w}) + (w(1))$$

and

$$w''(x) + (f(w(x)))' = 0.$$

It follows that  $w'(1) = g(w(1), \bar{w})$ . Since  $h(\sigma) > 0$ ,  $F(w; \alpha, \beta) = 0$  implies  $w(0) = 0$ .

**THEOREM 2.4B.** *Let  $f' > 0$  for  $u > 0$ . Suppose that  $w(x)$  is a positive solution of  $(B_1) \in C^2(0, 1) \cap C^1[0, 1]$ . Then  $w(x)$  satisfies*

$$G(w; \alpha, \beta) = \int_0^{w(x)} \frac{d\sigma}{g(\alpha, \beta) - f(\alpha) + f(\sigma)} = 1 - x$$

for  $0 \leq x \leq 1$  and  $g(\alpha, \beta) - f(\alpha) > 0$ . Here,  $\alpha = w(0)$  and  $\beta = \int_0^1 w(x) dx$ .

Conversely, if  $\alpha > 0$  satisfies  $G(\alpha; \alpha, \beta) = 1$ , and positive  $w(x)$  satisfies  $G(w; \alpha, \beta) = 1 - x$  with  $\int_0^1 w(x)dx = \beta$ , and  $g(\alpha, \beta) - f(\alpha) > 0$ , then  $w(x)$  is a positive solution of  $(B_1)$ .

PROOF: This result is proved much like the parallel proof above. However, we will include a sketch of the proof indicating where the condition  $g(\alpha, \beta) - f(\alpha) > 0$  is involved.

Let  $w(x)$  be a solution of  $(B_1)$ . Then a quadrature yields

$$g(\alpha, \beta) - f(\alpha) = -w'(x) - f(w(x))$$

on  $[0, 1]$ . In particular,  $g(\alpha, \beta) - f(\alpha) = -w'(1)$  which is positive by Lemma 2.1B.

Conversely, if  $g(\alpha, \beta) - f(\alpha) > 0$ , then  $g(\alpha, \beta) - f(\alpha) + f(\sigma)$  is positive for  $\sigma \in [0, \alpha]$ . Thus the proof can be proceeded in the same manner as that in Theorem 2.4A.

As an application of the above theorem, we characterize positive stationary solutions of Burgers' equation with  $f(u) = \frac{1}{2}\varepsilon u^2$  and  $g(u, v) = au^p v^q$  or  $g(u, v) = au^p v^q \pm \frac{1}{2}\varepsilon u^2$ , where  $a, \varepsilon > 0$ ,  $0 < p, q < \infty$ .

For problem  $(A_1)$  with  $g(u, v) = au^p v^q$ , let  $y = \int_0^x w(s)ds$ ,  $h(y) = u^2(x(y))$ , where  $x = x(y)$  is treated as a function of  $y$ . We obtain  $h''(y) + \varepsilon h'(y) = 0$  with  $h(0) = 0$ .

Therefore,  $h = C^2(1 - e^{-\varepsilon y})$  and  $w = C \left( 1 - e^{-\varepsilon \int_0^x w(s)ds} \right)^{1/2}$ . Let  $w(1) = \alpha$ ,

$$\bar{w} = \beta,$$

$$(2.1.1) \quad \alpha = C (1 - e^{-\varepsilon\beta})^{1/2}$$

$$(2.1.2) \quad w'(x) = \frac{1}{2} C \left( 1 - e^{-\varepsilon \int_0^x w(s) ds} \right)^{-1/2} e^{-\varepsilon \int_0^x w(s) ds} \cdot (\varepsilon w(x))$$

$$(2.1.3) \quad = \frac{1}{2} C^2 \varepsilon e^{-\varepsilon \int_0^x w(s) ds}.$$

Thus,

$$(2.2) \quad \frac{1}{2} C^2 \varepsilon e^{-\varepsilon\beta} = a \alpha^p \beta^q.$$

From (2.1.1) and (2.2), we obtain

$$(2.3) \quad \alpha^{p-2} = \frac{\varepsilon}{2a} (e^{\varepsilon\beta} - 1)^{-1} \beta^{-q}.$$

Hence,

$$(2.4) \quad F_1(\alpha, \beta) = \frac{1}{2} \varepsilon \alpha F(\alpha; \alpha, \beta) = \int_0^1 \frac{d\sigma}{\left(\frac{2a}{\varepsilon}\right) \alpha^{p-2} \beta^q + 1 - \sigma^2} = \frac{1}{2} \varepsilon \alpha.$$

On the other hand, the solution  $\tilde{w}(x)$  arising from (2.3) and (2.4) satisfies

$$\tilde{w}''(x) + f(\tilde{w}(x))' = 0 \quad 0 < x < 1,$$

$$\tilde{w}(0) = 0, \quad \tilde{w}'(1) = g(\alpha, \beta)$$

with  $\alpha = \tilde{w}(1)$ . Repeating an argument similar to that leading to (2.1)–(2.2), we have

$$\alpha^{p-2} = \frac{\varepsilon}{2a} \left( e^{\varepsilon\tilde{w}} - 1 \right)^{-1} \beta^{-q}.$$

Therefore by combining the above equation with (2.3) we find that  $\beta = \bar{\tilde{w}}$ .

If  $p = 2$ , (2.3) is equivalent to

$$(2.5) \quad Q(\beta) = \frac{2a}{\varepsilon} (e^{\varepsilon\beta} - 1) = \beta^{-q} = R(\beta),$$

which we now demonstrate to have exactly one solution.

$Q(0) = 0$ ,  $Q(+\infty) = +\infty$ , and  $Q'(\beta) > 0$ , while  $R(\beta) \rightarrow +\infty$  as  $\beta \rightarrow 0$ ,  $R(+\infty) = 0$  and  $R'(\beta) < 0$ . Moreover,  $F_1(\alpha, \beta) = F_1(\beta) = \int_0^1 \frac{d\sigma}{(\frac{2a}{\varepsilon})\beta^q + 1 - \sigma^2} = \frac{1}{2}\varepsilon\alpha$ . Therefore, only one  $w(1)$  exists for each  $\varepsilon$  and  $a$ . Thus, when  $p = 2$ , there is one solution of (A<sub>1</sub>) with this choice of  $g$ .

For  $p \neq 2$ , substituting (2.3) into (2.4) yields

$$(2.6) \quad \int_0^1 \frac{d\sigma}{(e^{\varepsilon\beta} - 1)^{-1} + 1 - \sigma^2} = \frac{\varepsilon}{2} \left( \frac{\varepsilon}{2a} \right)^{\frac{1}{p-2}} (e^{\varepsilon\beta} - 1)^{-\frac{1}{p-2}} \beta^{-\frac{q}{p-2}}.$$

Let  $\gamma^2 = (e^{\varepsilon\beta} - 1)^{-1} + 1$ ,  $\beta = \frac{1}{\varepsilon} \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right)$  on  $(1, \infty)$ ; and

$$\begin{aligned} \int_0^1 \frac{d\sigma}{\gamma^2 - \sigma^2} &= \varepsilon^{\frac{p+q-1}{p-2}} 2^{\frac{1-p}{p-2}} a^{\frac{1}{p-2}} (\gamma^2 - 1)^{\frac{1}{p-2}} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{\frac{q}{p-2}} \\ &= \delta (\gamma^2 - 1)^{\frac{1}{p-2}} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{\frac{q}{p-2}}, \end{aligned}$$

where  $\delta = \varepsilon^{\frac{p+q-1}{p-2}} 2^{\frac{1-p}{p-2}} a^{\frac{1}{p-2}}$ .

Set

$$(2.7) \quad \Phi(\gamma) = (\gamma^2 - 1)^{\frac{1}{p-2}} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{\frac{q}{p-2}} \cdot \int_0^1 \frac{d\sigma}{\gamma^2 - \sigma^2} = \delta.$$

We then obtain, on  $(1, \infty)$

$$\Phi(\gamma) = \frac{1}{2} (\gamma^2 - 1)^{\frac{1}{p-2}} \gamma^{-1} \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{\frac{q}{p-2}}.$$



If  $p > 2$ , set  $\lambda = \frac{1}{p-2}$ ,  $\mu = \frac{q}{p-2}$ ,

$$\Phi(\gamma) = \frac{1}{2} (\gamma^2 - 1)^{-\lambda} \gamma^{-1} \ln \left( \frac{\gamma+1}{\gamma-1} \right) \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^\mu,$$

and

$$\Phi'(\gamma) = \frac{1}{2} (\gamma^2 - 1)^{-\lambda-1} \gamma^{-2} \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^\mu K(\gamma),$$

where

$$K(\gamma) = [(-2\lambda - 1)\gamma^2 + 1] \ln \left( \frac{\gamma+1}{\gamma-1} \right) - 2\gamma - 2\mu \ln \left( \frac{\gamma+1}{\gamma-1} \right) / \ln \left( \frac{\gamma^2}{\gamma^2-1} \right).$$

Because  $K(\gamma) < 0$  on  $(1, \infty)$ , we have  $\Phi'(\gamma) < 0$ . We can easily see that  $\Phi(\gamma) \rightarrow +\infty$  as  $\gamma \rightarrow 1^+$  and that  $\Phi(+\infty) = 0$ . Therefore,  $(A_1)$  has one and only one positive solution for all  $\varepsilon > 0$ ,  $a > 0$ .

For  $p < 2$ , we first discuss the case  $p = 1$ .

$$\begin{aligned} \Phi(\gamma) &= \frac{1}{2} (\gamma^2 - 1) \gamma^{-1} \ln \left( \frac{\gamma+1}{\gamma-1} \right) \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^{-q} \\ &= \frac{1}{2} \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^{-q} \Phi_1(\gamma), \end{aligned}$$

where  $\Phi_1(\gamma) = (\gamma^2 - 1) \gamma^{-1} \ln \left( \frac{\gamma+1}{\gamma-1} \right)$ .

We thus find that

$$\Phi'(\gamma) = q \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^{-q-1} [\gamma(\gamma^2 - 1)]^{-1} \cdot \Phi_1(\gamma) + \frac{1}{2} \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^{-q} \Phi_1'(\gamma),$$

where

$$\Phi_1'(\gamma) = \gamma^{-2} [(\gamma^2 + 1) \ln((\gamma + 1)/(\gamma - 1)) - 2\gamma].$$

Letting  $\Phi_2(\gamma) = (\gamma^2 + 1) \ln((\gamma + 1)/(\gamma - 1)) - 2\gamma$ , we obtain

$$\Phi_2'(\gamma) = 2\gamma \ln((\gamma + 1)/(\gamma - 1)) - \frac{4\gamma^2}{\gamma^2 - 1}$$

and

$$\left(\frac{\Phi'_2(\gamma)}{2\gamma}\right)' = \frac{4}{(\gamma^2 - 1)^2} > 0,$$

because  $\frac{\Phi'_2(\gamma)}{2\gamma} \rightarrow 0$  as  $\gamma \rightarrow +\infty$ ,  $\Phi'_2(\gamma) < 0$  on  $(1, \infty)$ .  $\Phi_2(\gamma) \sim 2(\gamma^2 + 1)/(\gamma - 1) - 2\gamma$  as  $\gamma \rightarrow +\infty$ , it follows that  $\Phi_2(\gamma) > 0$  and, consequently, that  $\Phi'_1(\gamma) > 0$  on  $(1, \infty)$ , which shows  $\Phi'(\gamma) > 0$ . Clearly,  $\Phi(\gamma) \rightarrow 0$  as  $\gamma \rightarrow 1^+$ , and  $\Phi(+\infty) = +\infty$ . Hence, when  $p = 1$ , we obtain a unique positive solution.

Next, we examine the case  $1 < p < 2$ . Set

$$\Phi(\gamma) = \left\{ \frac{1}{2} (\gamma^2 - 1)^{\frac{p-1}{2-p}} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{-\frac{q}{2-p}} \right\} \Phi_1(\gamma)$$

where  $\Phi_1(\gamma)$  is the same as in the case  $p = 1$ . Obviously,  $\Phi'(\gamma) > 0$  for  $\gamma$  in  $(1, \infty)$ . Because  $\Phi(\gamma) \rightarrow 0$  as  $\gamma \rightarrow 1^+$  and  $\Phi(\gamma) \rightarrow +\infty$  as  $\gamma \rightarrow +\infty$ , we have exactly one positive solution in this case, as well.

Finally, we consider the case  $0 < p < 1$ . To simplify the discussion, we start with  $p + q \leq 1$ .

Set  $\lambda = \frac{1}{2-p}$ , and  $\mu = \frac{q}{2-p}$ ; then,  $\lambda > \frac{1}{2}$ , and  $\lambda + \mu \leq 1$ . We let

$$(2.8) \quad \Phi(\gamma) = \frac{1}{2} (\gamma^2 - 1)^\lambda \gamma^{-1} \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{-\mu}$$

and find that

$$\Phi'(\gamma) = \frac{1}{2} (\gamma^2 - 1)^{\lambda-1} \gamma^{-2} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{-\mu} \cdot K(\gamma),$$

where

$$K(\gamma) = [(2\lambda - 1)\gamma^2 + 1] \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) - 2\gamma + 2\mu \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) / \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right)$$

and

$$\begin{aligned}
\frac{1}{2}K'(\gamma) &= (2\lambda - 1)\gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{2\lambda\gamma^2}{\gamma^2-1} \\
&\quad - \frac{2\mu}{\gamma^2-1} \left[ \ln\left(\frac{\gamma^2}{\gamma^2-1}\right) - \gamma^{-1} \ln\left(\frac{\gamma+1}{\gamma-1}\right) \right] / \left[ \ln\left(\frac{\gamma^2}{\gamma^2-1}\right) \right]^2 \\
&= \left[ (2\lambda - 1)\gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{2(\lambda - \mu)\gamma^2}{\gamma^2-1} \right] - \left[ \frac{2\mu\gamma^2}{\gamma^2-1} - \frac{2\mu}{\gamma^2-1} / \ln\left(\frac{\gamma^2}{\gamma^2-1}\right) \right] \\
&\quad - \left\{ \frac{2\mu}{\gamma^2-1} \left[ 2\ln\left(\frac{\gamma^2}{\gamma^2-1}\right) - \gamma^{-1} \ln\left(\frac{\gamma+1}{\gamma-1}\right) \right] / \left[ \ln\left(\frac{\gamma^2}{\gamma^2-1}\right) \right]^2 \right\} \\
&= K_1(\gamma) - K_2(\gamma) - K_2(\gamma).
\end{aligned}$$

We claim that  $K_1(\gamma) < 0$  because  $\gamma \rightarrow +\infty$ ,  $\frac{K_1(\gamma)}{2\gamma} \sim \frac{(\lambda+\mu-1)}{\gamma} \leq 0$  and  $\left(\frac{K_1(\gamma)}{2\gamma}\right)' = -\frac{(2\lambda-1)}{\gamma^2-1} + (\lambda - \mu) \frac{\gamma^2+1}{(\gamma^2-1)^2} \geq \frac{1}{(\gamma^2-1)^2} [(1 - \lambda - \mu)\gamma^2 + 2(2\lambda - 1)] > 0$ .

$K_2(\gamma) \geq 0$  follows from the well-known inequality  $\ln(1+x) \geq \frac{x}{1+x}$ , with  $x$  replaced by  $\frac{1}{\gamma^2-1}$ .

To show that  $K_3(\gamma) > 0$ , we need only prove that

$$H(\gamma) = 2\gamma \ln\left(\frac{\gamma^2}{\gamma^2-1}\right) - \ln\left(\frac{\gamma+1}{\gamma-1}\right) > 0.$$

We have  $H'(\gamma) = 2\ln\left(\frac{\gamma^2}{\gamma^2-1}\right) - \frac{2}{\gamma^2-1}$  and  $H'(\gamma) \rightarrow 0$  as  $\gamma \rightarrow +\infty$ . Moreover,  $H''(\gamma) = \frac{4}{\gamma(\gamma^2-1)^2} > 0$ ; hence  $H'(\gamma) < 0$  on  $(1, \infty)$ . Note that  $H(\gamma) \rightarrow 0$  as  $\gamma \rightarrow +\infty$ , and therefore,  $H(\gamma) > 0$  for all  $\gamma > 1$ .

Thus,  $K'(\gamma) < 0$  on  $(1, \infty)$ .

If  $p + q = 1$ , which is equivalent to  $\lambda + \mu = 1$ , we find that  $\Phi'(\gamma) > 0$  on  $(1, \infty)$  because  $K(\gamma) \rightarrow 0$  as  $\gamma \rightarrow +\infty$ . We also observe that  $\lim_{\gamma \rightarrow 1^+} \Phi(\gamma) = 0$  and that  $\lim_{\gamma \rightarrow +\infty} \Phi(\gamma) = 1$ . Thus, if  $\delta \geq 1$ , that is, if  $a \geq 2^{1-p}$ , there is no solution, if  $\delta < 1$  ( $a < 2^{1-p}$ ), there is one.

For  $p + q < 1$  ( $\lambda + \mu < 1$ ),  $K(\gamma)$  becomes positive infinity as  $\gamma$  approaches 1, whereas  $K(\gamma) \sim 4(\lambda + \mu - 1)\gamma < 0$  as  $\gamma \rightarrow +\infty$ . Thus,  $K(\gamma)$  has exactly one sign

change, and  $\Phi(\gamma)$  increases before decreasing. We can also see that  $\lim_{\gamma \rightarrow 1^+} \Phi(\gamma) = 0$  and that  $\lim_{\gamma \rightarrow +\infty} \Phi(\gamma) = 0$ . Hence,  $(A_1)$  has none, one, or two positive solutions, depending on whether  $\delta > \Phi_M$ ,  $\delta = \Phi_M$ , or  $\delta < \Phi_M$ , where  $\Phi_M = \max_{1 < \gamma < \infty} \Phi(\gamma)$ . That is,

$$\varepsilon > (\Phi_M)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}}, \quad \varepsilon = (\Phi_M)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}},$$

or

$$\varepsilon < (\Phi_M)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}}.$$

The only remaining case is  $p + q > 1$  ( $\lambda + \mu > 1$ ). This time, let  $\Phi(\gamma) = \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{-\frac{p+q-1}{2-p}} \cdot \Phi_0(\gamma)$ , where  $\Phi_0(\gamma)$  is the same as  $\Phi(\gamma)$  for  $p + q = 1$ . We can easily check that  $\Phi'(\gamma) > 0$  for all  $\gamma$  in  $(1, \infty)$ , whereas  $\Phi(\gamma) \rightarrow 0$  as  $\gamma \rightarrow 1^+$ , and  $\Phi(\gamma) \rightarrow +\infty$  as  $\gamma \rightarrow +\infty$ . Thus, there is only one positive solution of  $(A_1)$  for all  $\varepsilon > 0$ ,  $a > 0$ . These results are summarized in Figs. 3.1 and 3.2.

For problem  $(B_1)$  with  $g(u, v) = au^p v^p$ , we have

$$h(y) = C^2 \left( 1 - e^{-\varepsilon(\beta-y)} \right),$$

$$w(x) = C \left( 1 - e^{-\varepsilon\beta + \varepsilon \int_0^x w(s) ds} \right),$$

and

$$w'(x) = -\frac{1}{2} C^2 \varepsilon e^{-\varepsilon\beta + \varepsilon \int_0^x w(s) ds}.$$

Consequently, we find the following equation:

$$(2.9) \quad \alpha^{p-2} = \frac{\varepsilon}{2a} (e^{\varepsilon\beta} - 1)^{-1} \beta^q,$$

which is the same as (2.3). We now need, however, an additional condition:

$$(2.10) \quad \frac{2a}{\varepsilon} \alpha^{p-2} \beta^q > 1.$$

Taking this condition into account, we obtain

$$(2.11) \quad G_1(\alpha, \beta) = \frac{1}{2} \varepsilon \alpha G(\alpha; \alpha, \beta) = \int_0^1 \frac{d\sigma}{\left(\frac{2a}{\varepsilon}\right) \alpha^{p-2} \beta^q - 1 + \sigma^2} = \frac{1}{2} \varepsilon \alpha.$$

For  $p = 2$ , let  $\beta^* = \left(\frac{\varepsilon}{2a}\right)^{1/q}$ , and set  $Q(\beta) = \frac{2a}{\varepsilon} \beta^q$ ,  $R(\beta) = (e^{\varepsilon\beta} - 1)^{-1}$ . We wish to solve  $Q(\beta) = R(\beta)$ . We have  $Q(\beta^*) = 1$ ,  $Q(+\infty) = +\infty$ , and  $Q'(\beta) > 0$  for  $\beta$  in  $(\beta^*, \infty)$ ;  $R(\beta) \rightarrow 0$  as  $\beta \rightarrow +\infty$  and if  $\varepsilon < (2a)^{1/(1+q)} (\ln 2)^{1/(1+q)}$ , then  $R(\beta^*) > 1$ . Clearly,  $R'(\beta) < 0$  on  $(\beta^*, \infty)$ . Thus, when  $\varepsilon < (2a)^{1/(1+q)} (\ln 2)^{1/(1+q)}$ , there is a unique solution of  $(B_1)$  for  $p = 2$ .

For  $p \neq 2$ , substitution of (2.9) in (2.11) yields

$$\int_0^1 \frac{d\delta}{(e^{\varepsilon\beta} - 1)^{-1} - 1 + \sigma^2} = \frac{\varepsilon}{2} \left(\frac{\varepsilon}{2a}\right)^{\frac{1}{p-2}} (e^{\varepsilon\beta} - 1)^{-\frac{1}{p-2}} \beta^{-\frac{q}{p-2}}.$$

Let  $\gamma^2 = (e^{\varepsilon\beta} - 1)^{-1} - 1$ . We then have

$$\int_0^1 \frac{d\sigma}{\gamma^2 + \sigma^2} = \delta (\gamma^2 + 1)^{\frac{1}{p-2}} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{\frac{q}{p-2}} \quad \text{on } (0, \infty),$$

where  $\delta = e^{\frac{p+q-1}{p-2}} 2^{\frac{1-p}{p-2}} a^{\frac{1}{p-2}}$ .

Set

$$\Psi(\gamma) = (\gamma^2 + 1)^{\frac{1}{p-2}} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{\frac{q}{p-2}} \int_0^1 \frac{d\sigma}{\gamma^2 + \sigma^2} = \delta.$$

We then find that

$$(2.13) \quad \Psi(\gamma) = \tan^{-1} \frac{1}{\gamma} \gamma^{-1} (\gamma^2 + 1)^{\frac{1}{p-2}} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\frac{q}{p-2}}.$$

If  $p > 2$ , set  $\lambda = \frac{1}{p-2}$ ,  $\mu = \frac{q}{p-2}$ ,

$$\Psi(\gamma) = \tan^{-1} \frac{1}{\gamma} \gamma^{-1} (\gamma^2 + 1)^{-\lambda} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^\mu,$$

and

$$\Psi'(\gamma) = \tan^{-1} \frac{1}{\gamma} (\gamma^2 + 1)^{-\lambda-1} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^\mu J(\gamma),$$

where

$$J(\gamma) = -2\lambda - \gamma^{-2}(\gamma^2 + 1) - \gamma^{-1} / \tan^{-1} \frac{1}{\gamma} - \frac{2\mu}{\gamma^2 + 2} / \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right).$$

Since  $J(\gamma) < 0$  on  $(0, \infty)$ , it follows that  $\Psi'(\gamma) < 0$ .

We see that  $\Psi(\gamma) \rightarrow +\infty$  as  $\gamma \rightarrow 0^+$  and that  $\Psi(+\infty) = 0$ . Hence,  $(B_1)$  has one and only one positive solution for each  $\varepsilon > 0$ ,  $a > 0$ .

In the case of  $p < 2$ , let  $\lambda = \frac{1}{2-p}$ , and  $\mu = \frac{q}{2-p}$ . We thus obtain

$$(2.14) \quad \Psi(\gamma) = \tan^{-1} \frac{1}{\gamma} \gamma^{-1} (\gamma^2 + 1)^\lambda \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\mu}$$

and

$$\Psi'(\gamma) = \tan^{-1} \frac{1}{\gamma} (\gamma^2 + 1)^{\lambda-1} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\mu} J(\gamma),$$

where

$$J(\gamma) = 2\lambda - \gamma^{-2}(\gamma^2 + 1) - \gamma^{-1} / \tan^{-1} \frac{1}{\gamma} + \frac{2\mu}{\gamma^2 + 2} / \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right)$$

and

$$\begin{aligned} J'(\gamma) &= 2\gamma^{-3} + \left[ \gamma^{-2} / \tan^{-1} \frac{1}{\gamma} - \frac{1}{\gamma(\gamma^2 + 1)} \right] / \left( \tan^{-1} \frac{1}{\gamma} \right)^2 \\ &\quad + \left\{ \frac{4\mu\gamma}{(\gamma^2 + 2)^2(\gamma^2 + 1)} / \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^2 - \frac{4\mu\gamma}{(\gamma^2 + 2)^2} / \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right\} \\ &= 2\gamma^{-3} + J_1(\gamma) + J_2(\gamma). \end{aligned}$$

Using  $\tan^{-1} x \geq \frac{x}{x^2+1}$  with  $x = \gamma^{-1}$ , we find that  $J_1(\gamma) \geq 0$ . We have  $J_2(\gamma) \geq 0$  since  $\ln(1+x) \leq x$  with  $x = \frac{1}{\gamma^2+1}$ . Thus,  $J'(\gamma) > 0$  on  $(0, \infty)$ .

Since  $1 \leq p < 2$  corresponds to  $1 \leq \lambda < \infty$ , we observe that  $J(\gamma) \rightarrow -\infty$  as  $\gamma \rightarrow 0^+$  and that  $J(\gamma) \sim 2(\lambda + \mu + 1) > 0$  as  $\gamma \rightarrow +\infty$ . Hence  $J(\gamma)$  changes sign only once, and it follows that none, one or two positive solutions of  $(B_1)$  exist, according to whether

$$\varepsilon < (\Psi_m)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}}, \quad \varepsilon = (\Psi_m)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}},$$

or

$$\varepsilon > (\Psi_m)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}},$$

where  $\Psi_m = \min_{0 < \gamma < \infty} \Psi(\gamma)$ .

For  $0 < p < 1$ ,  $\frac{1}{2} < \lambda < 1$ . If  $\lambda + \mu < 1$ , i.e., if  $p + q < 1$ ,  $J(\gamma) \sim 2(\lambda + \mu - 1) < 0$  as  $\gamma \rightarrow +\infty$ , so  $\Psi'(\gamma) < 0$  for any  $\gamma$  in  $(0, \infty)$ . In constrast,  $\lim_{\gamma \rightarrow 0^+} \Psi(\gamma) = +\infty$  and  $\lim_{\gamma \rightarrow +\infty} \Psi(\gamma) = 0$ . Therefore,  $(B_1)$  has only one solution.

If  $\lambda + \mu = 1$  ( $p + q = 1$ ),  $J(\gamma) \rightarrow 0$  as  $\gamma \rightarrow +\infty$ , it follows that  $\Psi'(\gamma) < 0$  for  $\gamma > 0$ .  $\Psi(\gamma) \rightarrow +\infty$  as  $\gamma \rightarrow 0^+$  and  $\Psi(\gamma) \rightarrow 1$  as  $\gamma \rightarrow +\infty$ , however. Thus, there is no solution if  $\delta \leq 1$  ( $a \leq 2^{1-p}$ ), and there is a unique solution if  $\delta > 1$  ( $a > 2^{1-p}$ ).

If  $\lambda + \mu > 1$  ( $p + q > 1$ ),  $J(\gamma) \sim 2(\lambda + \mu - 1) > 0$  as  $\gamma \rightarrow +\infty$ , but  $J(\gamma) \rightarrow -\infty$  as  $\gamma \rightarrow 0^+$ .  $\Psi'(\gamma)$  thus has only one sign change; and  $\Psi(\gamma)$  decreases before increasing with  $\lim_{\gamma \rightarrow 0^+} \Psi(\gamma) = +\infty$  and  $\lim_{\gamma \rightarrow +\infty} \Psi(\gamma) = +\infty$ . For this reason,  $(B_1)$  has none, one, or two positive solutions according to whether

$$\varepsilon < (\Psi_m)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}}, \quad \varepsilon = (\Psi_m)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}},$$

or

$$\varepsilon > (\Psi_m)^{\frac{2-p}{1-p-q}} 2^{\frac{1-p}{1-p-q}} a^{\frac{1}{p+q-1}}.$$

The solution diagrams then have the form indicated in Figs. 3.3 and 3.4.

Next, we will discuss the problems with  $f(u) = \frac{1}{2}\varepsilon u^2$  and  $g(u, v) = au^p v^q - \frac{1}{2}\varepsilon u^2$  for  $(A_1)$  or  $g(u, v) = au^p v^q + \frac{1}{2}\varepsilon u^2$  for  $(B_1)$ .

For  $(A_1)$ , we first obtain

$$(2.15) \quad \alpha^{p-2} = \left(\frac{\varepsilon}{2a}\right) (1 - e^{-\varepsilon\beta})^{-1} \beta^{-q}.$$

Let  $\gamma^2 = (1 - e^{-\varepsilon\beta})^{-1}$ , then  $\beta = \frac{1}{\varepsilon} \ln \left(\frac{\gamma^2}{\gamma^2 - 1}\right)$ , and on  $(1, \infty)$ ,

$$(2.16) \quad \Phi(\gamma) = \frac{1}{2} \gamma^{\frac{2}{2-p}-1} \ln \left(\frac{\gamma+1}{\gamma-1}\right) \left[ \ln \left(\frac{\gamma^2}{\gamma^2 - 1}\right) \right]^{\frac{1}{p-2}} = \delta,$$

where  $\delta$  is the same as in (2.7). Notice that  $\Phi(\gamma)$  is not the same as in (2.7).

If  $p > 2$ , calculations show that  $\Phi'(\gamma) < 0$ ,  $\Phi(1^+) = +\infty$ , and  $\Phi(+\infty) = 0$ ; hence, only one solution exists.

The conclusion for  $p = 2$  is the same. We only need to show that

$$Q(\beta) = \frac{2a}{\varepsilon} (1 - e^{-\varepsilon\beta}) = \beta^{-q} = R(\beta)$$

has a unique solution.

For  $0 < p < 2$ , we have

$$(2.17) \quad \Phi(\gamma) = \frac{1}{2} \gamma^{2\lambda-1} \ln \left(\frac{\gamma+1}{\gamma-1}\right) \left[ \ln \left(\frac{\gamma^2}{\gamma^2 - 1}\right) \right]^{-\mu},$$

where  $\lambda = \frac{1}{2-p}$ ,  $\mu = \frac{q}{2-p}$ .

$$\Phi'(\gamma) = \frac{1}{2} (\gamma^2 - 1)^{-1} \gamma^{2\lambda-2} \ln \left(\frac{\gamma+1}{\gamma-1}\right) \left[ \ln \left(\frac{\gamma^2}{\gamma^2 - 1}\right) \right]^{-\mu} K(\gamma),$$

with  $K(\gamma)$  having the following form:

$$K(\gamma) = (2\lambda - 1)(\gamma^2 - 1) - 2\gamma \left/ \ln \left(\frac{\gamma+1}{\gamma-1}\right) \right. + 2\mu \left/ \ln \left(\frac{\gamma^2}{\gamma^2 - 1}\right) \right.$$



Because  $2\lambda - 1 > 0$  and  $\gamma \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \leq \ln \left( \frac{\gamma + 1}{\gamma - 1} \right)$  on  $(1, \infty)$ ,  $K(\gamma) > 0$  if  $\mu \geq 1$ —which implies that  $p + q \geq 2$ . For  $p + q > 2$ , we find that  $\Phi(\gamma) \rightarrow 0$  as  $\gamma \rightarrow 1^+$ , and  $\Phi(\gamma) \rightarrow +\infty$  as  $\gamma$  tends to infinity, there is thus only one solution. If  $p + q = 2$ ,  $\lim_{\gamma \rightarrow 1^+} \Phi(\gamma) = \frac{1}{2}$  and  $\lim_{\gamma \rightarrow +\infty} \Phi(\gamma) = +\infty$ , then, for  $\delta \leq \frac{1}{2}$ ,  $(A_1)$  has no solution; but if  $\delta > \frac{1}{2}$ , one solution.

If  $p + q \leq 1$ , since  $\gamma^2 - 1 \leq 1 / \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right)$ , we find that

$$\begin{aligned} K(\gamma) &\leq (2\lambda - 1 + 2\mu) / \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) - 2\gamma / \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) \\ &\leq 1 / \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) - 2\gamma / \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) \\ &< 0. \end{aligned}$$

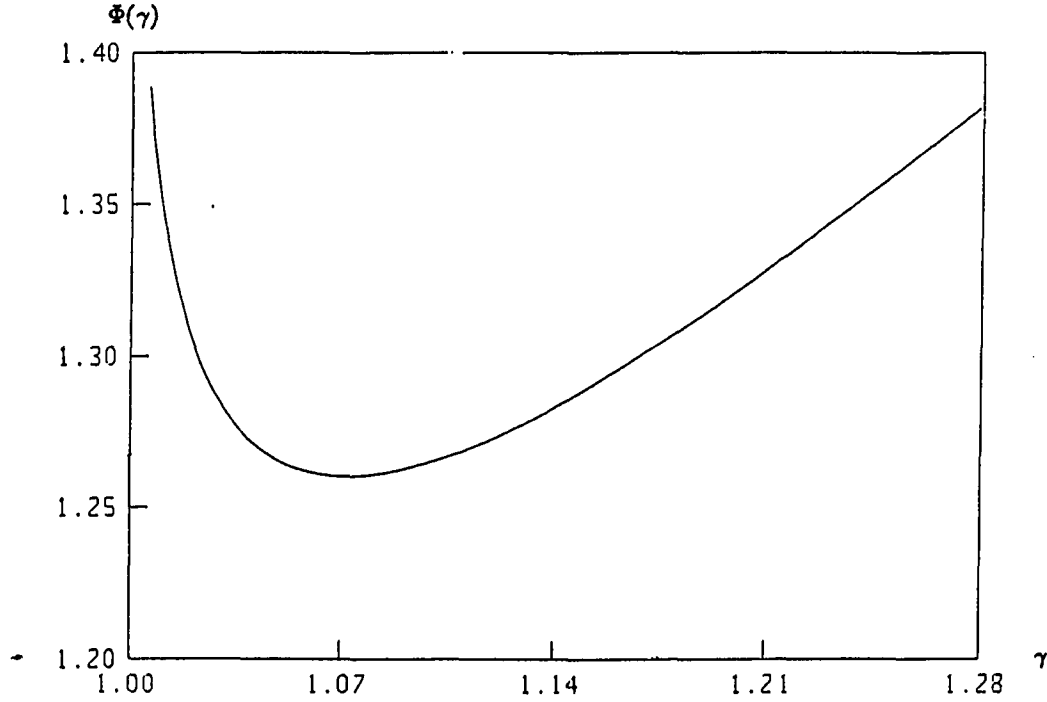


Figure 1:  $\Phi(\gamma)$  for  $g(u, v) = au^p v^q - \frac{1}{2}\varepsilon u^2$ ,  
 $a > 0$ ,  $p = 1$ ,  $q = 0.5$ .

For  $p + q = 1$ ,  $\Phi(\gamma) \rightarrow \infty$  as  $\gamma \rightarrow 1$  and  $\Phi(\gamma) \rightarrow 1$  as  $\gamma \rightarrow +\infty$ , thus there is one solution if  $\delta > 1$  but none if  $\delta \leq 1$ ; whereas for  $p + q < 1$ ,  $\Phi(\gamma) \rightarrow \infty$  as  $\gamma \rightarrow 1$  and  $\Phi(\gamma) \rightarrow 0$  as  $\gamma \rightarrow +\infty$ , then there is a unique solution.

The case  $1 < p + q < 2$  is not so amenable to analysis, but bearing numerical evidence in mind (see Fig. 1), we may conjecture as follows: Because  $\Phi(\gamma) \rightarrow \infty$  as  $\gamma \rightarrow 1$  or  $+\infty$ , there is none, one, or two solutions.

See Figs. 4.1, 4.2 for the solution diagrams.

Next, for  $(B_1)$ , we have the equation

$$(2.18) \quad \alpha^{p-2} = \frac{\varepsilon}{2a}(2 - e^{\varepsilon\beta})(e^{\varepsilon\beta} - 1)^{-1}\beta^{-q}.$$

Setting  $(2 - e^{\varepsilon\beta})(e^{\varepsilon\beta} - 1)^{-1} = \gamma^2$  leads to  $\beta = \frac{1}{\varepsilon} \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right)$ . Define

$$(2.19) \quad \Psi(\gamma) = \gamma^{\frac{2}{2-p}-1} \tan^{-1} \frac{1}{\gamma} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{\frac{2}{p-2}} = \delta \quad \text{on } (0, \infty).$$

For  $p > 2$ , calculations show that  $\Psi'(\gamma) < 0$ ,  $\Psi(0^+) = +\infty$ , and  $\Psi(+\infty) = 0$ . There is thus a unique solution. The same result holds for  $p = 2$  because the equation  $Q(\beta) = \frac{2a}{\varepsilon}\beta^q = (2 - e^{\varepsilon\beta})(e^{\varepsilon\beta} - 1)^{-1} = R(\beta)$  has exactly one solution ( $Q' > 0$ ,  $Q(0^+) = 0$ ,  $Q(\infty) = +\infty$ ;  $R' < 0$ ,  $R(0^+) = +\infty$ ,  $R(\infty) = -1$ .)

If  $1 \leq p < 2$ , we have, with  $\lambda = \frac{1}{2-p}$ ,  $\mu = \frac{q}{2-p}$ ,

$$(2.20) \quad \Psi(\gamma) = \gamma^{2\lambda-1} \tan^{-1} \frac{1}{\gamma} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\mu}$$

and

$$\Psi'(\gamma) = \gamma^{2\lambda}(\gamma^2 + 1)^{-1} \tan^{-1} \frac{1}{\gamma} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\mu} J(\gamma),$$

where

$$J(\gamma) = \frac{(2\lambda - 1)(\gamma^2 + 1)}{\gamma^2} - \frac{1}{\left( \gamma \tan^{-1} \frac{1}{\gamma} \right)} + 2\mu \left[ (\gamma^2 + 2) \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-1}.$$

Note that  $2\lambda - 1 \geq 1$ . From the inequality  $(\gamma^2 + 1) \tan^{-1} \frac{1}{\gamma} \geq \gamma$ , we see that  $J(\gamma) > 0$ . It follows that  $\Psi'(\gamma) > 0$  on  $(0, \infty)$ . Since  $\Psi(0) = 0$  and  $\Psi(+\infty) = +\infty$ , there is exactly one positive solution.

Although we cannot provide a rigorous analysis in case  $0 < p < 1$ , numerical results (see Fig. 2.1 – Fig. 2.3) support the following conjectures:

- i) When  $p + q > 1$ ,  $\Psi(0) = 0$ ,  $\Psi(+\infty) = +\infty$ , one solution may be obtained.
- ii) When  $p + q = 1$ ,  $\Psi(0) = 0$ ,  $\lim_{\gamma \rightarrow +\infty} \Psi(\gamma) = 1$ , no solution may be obtained if  $\delta \geq 1$ , and one solution may be obtained if  $\delta < 1$ .
- iii) When  $p + q < 1$ ,  $\Psi(0) = \Psi(+\infty) = 0$  indicates that none, one, or two solutions may be obtained.

Solution diagrams based, in part, on the numerical experiments are given in Figs. 4.3 and 4.4.

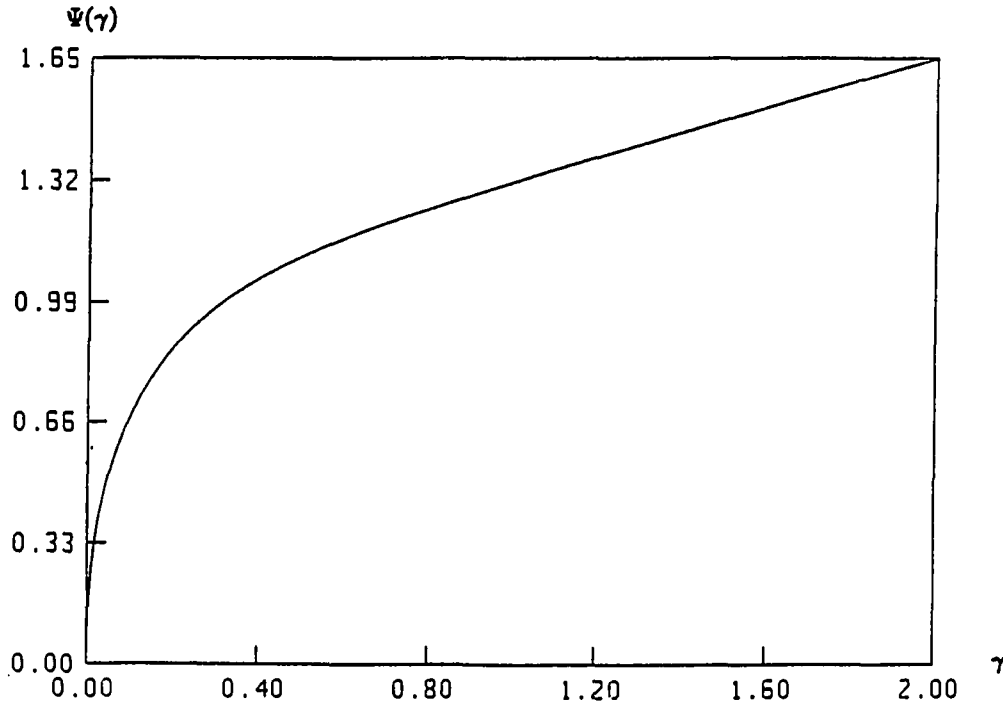


Figure 2.1:  $\Psi(\gamma)$  for  $g(u, v) = au^p v^q + \frac{1}{2}\varepsilon u^2$ ,  
 $a > 0$ ,  $p = 0.6$ ,  $q = 0.8$ .

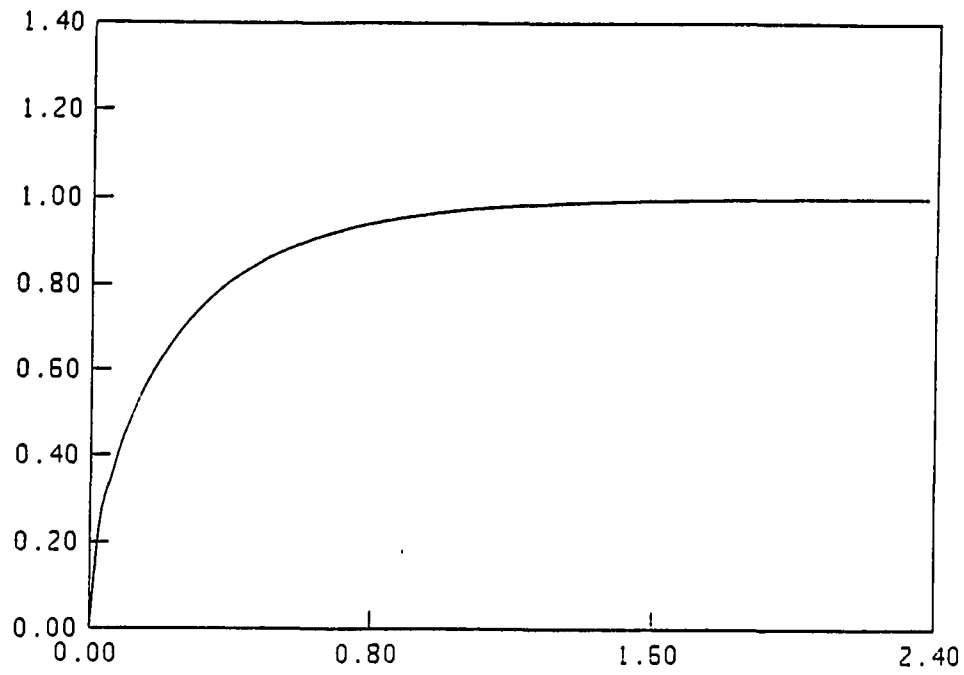


Figure 2.2: Same as for Fig. 2.1 except  $p = 0.7, q = 0.3$ .

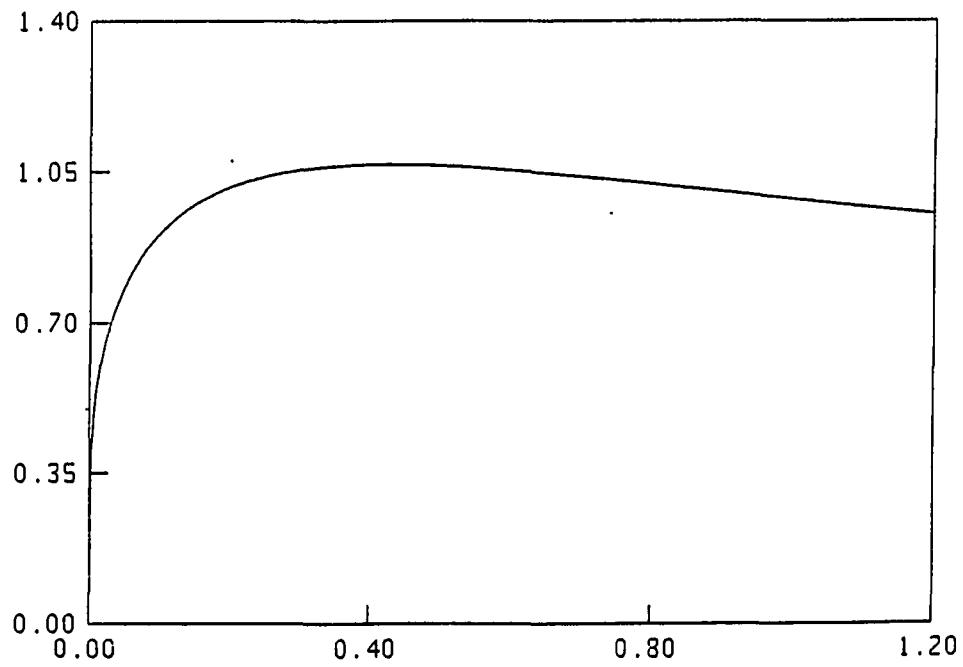


Figure 2.3: Same as for Fig. 2.1 except  $p = 0.4, q = 0.4$ .

Finally, for  $g(u, v) = au^p v^q$  or  $g(u, v) = au^p v^q \pm \frac{1}{2}\epsilon u^2$ , if  $a < 0$ , we show that the null solution is the only nonnegative solution of  $(A_1)$  or  $(B_1)$ .

For  $(A_1)$ , by Lemma 2.1A, positive solution must satisfy  $w'(1) > 0$ , which is impossible for  $a < 0$ . Hence  $w(x) \equiv 0$ .

For  $(B_1)$ , from the condition  $g(\alpha, \beta) - f(\alpha) > 0$  in Theorem 2.4B, we see that no positive solution can exist. Hence  $w(x)$  must be trivial.

### 3. Comparison Theorem, Local Existence, and Blow Up of Solutions

Let  $D_T = (0, 1) \times (0, T)$  and  $D_T \cup \Gamma_T = [0, 1] \times [0, T]$ ; by a solution  $u(x, t)$  of (A) or (B), we mean that  $u(x, t)$  is continuous in  $D_T \cup \Gamma_T$ , twice continuously differentiable in  $x$  and once in  $t$  on  $D_T$ . As is well known, comparison theorems are a powerful tool for studying equations of parabolic type. For parabolic problems with nonlocal nonlinearities, however, there are no known comparison theorems, as there are when the nonlinear terms are local. For this reason, we attempt to establish such a theory at the beginning of this section.

First, we proceed to the definitions of subsolution and supersolution of (A) and (B).

**DEFINITION A:** A function  $u(x, t)$  is called a *subsolution* of (A) on  $D_T$  if  $u \in C^{2,1}(D_T) \cap C(D_T \cup \Gamma_T)$ , satisfying

$$(A') \quad \begin{aligned} u_t &\leq u_{xx} + (f(u))_x & 0 < x < 1, \quad 0 < t < T, \\ u(0, t) &\leq 0 & 0 < t < T, \\ u_x(1, t) &\leq g(u(1, t), \bar{u}) & 0 < t < T, \\ u(x, 0) &\leq u_0(x) & 0 \leq x \leq 1. \end{aligned}$$

A supersolution is defined by  $(A')$  with each " $\leq$ " replaced by " $\geq$ ".

**DEFINITION B:** A function  $u(x, t)$  is called a *subsolution* of (B) on  $D_T$  if  $u \in C^{2,1}(D_T) \cap C(D_T \cup \Gamma_T)$ , satisfying

$$(B') \quad \begin{aligned} u_t &\leq u_{xx} + (f(u))_x & 0 < x < 1, \quad 0 < t < T, \\ -u_x(0, t) &\leq g(u, (0, t), \bar{u}) & 0 < t < T, \\ u(1, t) &\leq 0 & 0 < t < T, \\ u(x, 0) &\leq u_0(x) & 0 \leq x \leq 1. \end{aligned}$$

A supersolution is defined by (B') with each " $\leq$ " replaced by " $\geq$ ".

**THEOREM 3.1.** Suppose that  $f$  is continuously differentiable,  $g_\xi(\xi, \eta)$  is continuous for  $\xi \geq 0$ ,  $\eta \geq 0$ , and  $g_\eta(\xi, \eta) \geq 0$  for  $\xi \geq 0$ ,  $\eta \geq 0$ . Let  $u$  and  $v$  be a nonnegative supersolution and a nonnegative subsolution, respectively, of (A) or (B), with  $u(x, 0) \geq v(x, 0)$  for  $x \in (0, 1)$ . Then  $u \geq v$  in  $D_T \cup \Gamma_T$ .

**PROOF:** We prove this for (A) only. A similar argument holds for (B). For every  $t \in [0, T)$  and every nonnegative  $\varphi(x, t) \in C^{2,1}(\bar{D}_T)$  with  $\varphi(0, t) = 0$ , the subsolution  $v$  satisfies the following integral inequality:

$$\begin{aligned} \int_0^1 v(x, t) \varphi(x, t) dx &\leq \int_0^1 v_0(x) \varphi(x, 0) dx \\ &\quad + \int_0^t \int_0^1 [v \varphi_\tau - (v_x + f(v)) \varphi_x] dx d\tau \\ &\quad + \int_0^t [g(v(1, \tau), \bar{v}) + f(v(1, \tau))] \varphi(1, \tau) d\tau. \end{aligned}$$

The supersolution  $u$  satisfies the above with reversed inequality.

We integrate by parts in both the above inequality and that satisfied by  $u$ , and

subtract the two resultant expressions, then

$$\begin{aligned}
 (*) \quad & \int_0^1 (v(x, t) - u(x, t)) \varphi(x, t) dx \\
 & \leq \int_0^1 (v(x, 0) - u(x, 0)) \varphi(x, 0) dx \\
 & \quad + \int_0^t \int_0^1 (v - u) (\varphi_\tau + \varphi_{xx} - A(x, \tau) \varphi_x) dx d\tau \\
 & \quad + \int_0^t (v - u)(1, \tau) [B(1, \tau) \varphi(1, \tau) - \varphi_x(1, \tau)] d\tau \\
 & \quad + \int_0^t C(1, \tau) \varphi(1, \tau) \int_0^1 (v - u) dx d\tau \\
 & \quad + \int_0^t [v(0, \tau) - u(0, \tau)] \varphi_x(0, \tau) d\tau,
 \end{aligned}$$

where

$$\begin{aligned}
 A(x, t) &= f'(\theta_1(x, t)), \\
 B(x, t) &= g_\xi(\theta_2(x, t), \bar{v}) + f'(\theta_3(x, t)), \\
 C(x, t) &= g_\eta(u, \theta_4(x, t)),
 \end{aligned}$$

with  $\theta_i$  ( $i = 1, 4$ ) between  $u$  and  $v$ .

Note that by the hypotheses for  $f$  and  $g$ ,  $A$ ,  $B$ , and  $C$  are bounded on  $\overline{D}_T$  in the uniform norm. We denote the bound by  $M_0$ .

Now we define two sequences  $\{A_n\}$  and  $\{B_n\}$  in such a way that

- i)  $A_n, B_n \in C^\infty(\overline{D}_T)$ ,
- ii)  $|A_n| \leq M_0, |B_n| \leq M_0$ ,
- iii)  $A_n \rightarrow A, B_n \rightarrow B$  as  $n \rightarrow \infty$  in  $D_T$ ,

and set up a backward problem on  $D_t$ :

$$\begin{aligned}
 (A^*) \quad & \varphi_{n\tau} + \varphi_{nxx} - A_n \varphi_{nx} = 0 & 0 < x < 1, \quad 0 < \tau < t, \\
 & \varphi_n(0, \tau) = 0 & 0 < \tau < t, \\
 & \varphi_{nx}(1, \tau) = B_n \varphi_n & 0 < \tau < t, \\
 & \varphi_n(x, t) = \chi(x) & 0 \leq x \leq 1.
 \end{aligned}$$

Here,  $\chi(x) \in C_0^\infty(0, 1)$ ,  $0 \leq \chi \leq 1$ .

The existence of  $\varphi_n \in C^{2,1}(\overline{D}_T)$  follows from the fact that by variable transformation  $s = t - \tau$ ,  $(A^*)$  can be rewritten into

$$\begin{aligned}
 (A^{**}) \quad & \psi_{ns} - \psi_{nxx} + A_n \psi_{nx} = 0 & 0 < x < 1, \quad 0 < s < t, \\
 & \psi_n(0, s) = 0 & 0 < s < t, \\
 & \psi_{nx}(1, s) = B_n \psi_n & 0 < s < t, \\
 & \psi_n(x, 0) = \chi(x) & 0 \leq x \leq 1.
 \end{aligned}$$

Recalling Theorem 3 in [11], we find that  $\varphi = \lim_{n \rightarrow \infty} \varphi_n$  is a solution of  $(A^*)$  with  $A_n, B_n$  replaced by  $A, B$ , and  $\varphi \in C^{2,1}(\overline{D}_T)$ . Since  $\varphi_n(0, \tau) = 0$ ,  $\varphi_n(x, t) \geq 0$ , we have that  $\varphi_n \geq 0$  and  $\varphi_{nx}(0, \tau) \geq 0$ , which implies that  $\varphi \geq 0$ ,  $\varphi_x(0, \tau) \geq 0$ .

Substituting  $\varphi$  in  $(*)$  yields

$$\begin{aligned}
 \int_0^1 (v(x, t) - u(x, t)) \chi(x) dx &\leq M_1 \int_0^1 (v(x, 0) - u(x, 0))^+ dx \\
 &+ \int_0^t C(1, \tau) \varphi(1, \tau) \int_0^1 (v - u) dx d\tau,
 \end{aligned}$$

where  $M_1 = \sup_{\overline{D}_T} |\varphi|$ .

Since this inequality holds for every  $\chi$ , we can choose a sequence  $\{\chi_n\}$  on  $(0, 1)$  converging to

$$\chi = \begin{cases} 1 & \text{if } v(x, t) - u(x, t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$



Meanwhile, noting that  $C \geq 0$ ,  $v - u = (v - u)^+ - (v - u)^-$ , we find that

$$\begin{aligned} \int_0^1 (v(x, t) - u(x, t))^+ dx &\leq M_1 \int_0^1 (v(x, 0) - u(x, 0))^+ dx \\ &\quad + M_2 \int_0^t \int_0^1 (v(x, \tau) - u(x, \tau))^+ dx d\tau, \end{aligned}$$

which leads to, by Gronwall's inequality,

$$\int_0^1 (v(x, t) - u(x, t))^+ dx \leq M_1 (1 + e^{M_1 t}) \int_0^1 (v(x, 0) - u(x, 0))^+ dx,$$

where  $M_2 = M_0 M_1$ .

Thus, the conclusion follows from the condition on initial data.

**REMARK 3.1:** For nonnegative subsolution and supersolution, if one of them is positive at  $x = 1$  for (A) or  $x = 0$  for (B), then the conditions on  $g$  can be relaxed as follows:  $g_\xi(\xi, \eta)$  is continuous for  $\xi > 0$ ,  $\eta > 0$ , and  $g_\eta(\xi, \eta) \geq 0$  for  $\xi > 0$ ,  $\eta > 0$ .

By a similar argument as above, we can show the positivity of solutions with nonnegative initial values.

**THEOREM 3.2.** *Let  $f$  be continuously differentiable, and suppose that  $g_\xi(\xi, \eta)$  is continuous in a neighborhood of  $\xi = 0$ . If  $u_0(x) \geq 0$  on  $[0, 1]$ , then the solution of (A) or (B) is positive in  $D_T \cup \Gamma_T$ , except at  $x = 0$  or  $x = 1$ .*

**PROOF:** Obviously  $v \equiv 0$  is a subsolution. Note that this time in (\*) the integral  $\int_0^t C(1, \tau) \varphi(1, \tau) \int_0^1 (v - u) dx d\tau$  does not appear. Hence  $u$  is nonnegative, and its positivity follows from strong maximum principle.

We also have the following monotonicity result:

**COROLLARY 3.3.** *If  $u_0(x) \geq 0$  and  $u(x, t) \geq u_0(x)$  ( $\leq u_0(x)$ ) in  $D_T \cup \Gamma_T$  for (A) or (B), then  $u_t(x, t) \geq 0$  ( $\leq 0$ ) in  $D_T$ .*

**PROOF:** Let  $v(x, t) = u(x, t + h)$  ( $h > 0$ ).  $v$  is a supersolution (subsolution) of (A) or (B), and therefore  $u(x, t + h) \geq u(x, t)$  ( $\leq u(x, t)$ ). Since  $h$  is arbitrary,  $u$  is increasing (decreasing) in  $t$  for fixed  $x$ , and hence  $u_t \geq 0$  ( $\leq 0$ ).

Sometimes the condition above is not easy to check. By requiring another restriction on the initial datum, we have the same conclusion.

**COROLLARY 3.4.** *Suppose that the hypotheses for  $f$  and  $g$  in Theorem 3.1 are satisfied. If  $u_0'' + (f(u_0))' \geq 0$  ( $\leq 0$ ) on  $[0, 1]$  for (A) or (B), then  $u_t(x, t) \geq 0$  ( $\leq 0$ ) in  $D_T$ .*

**PROOF:** We prove the first statement for (A) only.

Let  $\{u_0^{(n)}\}$  be a sequence monotonely approximating  $u_0(x)$  from below, and let  $\{u^{(n)}\}$  be the corresponding solutions with the corner compatibility condition satisfied, that is,  $u^{(n)}(0, 0) = 0$ , and  $u_x^{(n)}(1, 0) = g(u^{(n)}(1, 0), \bar{u}_0^{(n)})$ . Then, by Theorem 7.4 in [7, Chap. V], the functions  $u_i^{(n)}$  are continuous up to the parabolic boundary.

Let  $v^{(n)} = u_i^{(n)}$ . We then have

$$\begin{aligned} v_t^{(n)} &= v_{xx}^{(n)} + f'(u^{(n)})v_x^{(n)} + u_x^{(n)}f''(u^{(n)})v^{(n)} & 0 < x < 1, \quad 0 < t < T, \\ v^{(n)}(0, t) &= 0 & 0 < t < T, \\ v_x^{(n)}(1, t) &= g_u(u^{(n)}, \bar{u}^{(n)})v^{(n)} + g_{\bar{u}}(u^{(n)}, \bar{u}^{(n)})\bar{v}^{(n)} & 0 < t < T, \\ v^{(n)}(x, 0) &\geq 0 & 0 \leq x \leq 1. \end{aligned}$$

Following the same procedure as in Theorem 3.1, we obtain  $v^{(n)} \geq 0$  on  $D_T$ . Hence, by taking the limit, we see that  $u_t \geq 0$  on  $D_T$ .

Next, we turn our attention to the local existence theorem for the solutions, which we shall state for problem (A) only although it is also true for (B).

To prove the solvability of (A), we need an a priori estimate for  $\sup_{\overline{D_T}} u(x, t)$ .

LEMMA 3.5. *Let  $u(x, t)$  be the solution of the following problem:*

$$(C) \quad \begin{aligned} u_t &= u_{xx} + (f(u))_x & 0 < x < 1, \quad 0 < t < T, \\ u(0, t) &= 0 & 0 < t < T, \\ u_x(1, t) &= h(1, t) & 0 < t < T, \\ u(x, 0) &= u_0(x) \geq 0 & 0 \leq x \leq 1. \end{aligned}$$

Then for any  $M > 0$ , with  $\max_{0 \leq x \leq 1} u_0(x) < \frac{M}{2}$ , there is a  $T_0$ , so that  $\sup_{\overline{D_{T_0}}} u(x, t) \leq M$ .

PROOF: Let  $f_{1M} = \sup_{|\xi| \leq M} |f'(\xi)|$ ,  $h_M = \sup_{0 \leq t \leq T} |h(1, t)|$ , and set

$$w(x) = \delta \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{\pi(x-1)}{\delta} \right) \right],$$

where  $0 < \delta < M$  and  $w(x)$  is taken from [9].

A straightforward computation shows that  $0 < w(x) < \delta$ ,  $0 < w'(x) \leq 1$ ,  $0 \leq w''(x) \leq \frac{3}{\delta}$  in  $[0, 1]$  and  $w'(1) = 1$ .

Choose  $\delta = \min \left\{ M, \frac{1}{f_{1M}}, \frac{M}{4h_M} \right\}$  and let  $v(x, t) = \frac{4h_M}{\delta}t + h_M w(x) + \frac{M}{2}$ . Then in  $D_T$  with  $T_0 = \frac{\delta M}{16h_M}$ ,  $v(x, t) \leq M$  and  $v(x)$  is a supersolution of (C). By a conventional comparison theorem, we see that  $u(x, t) \leq v(x, t) \leq M$ .

We now adopt the monotone iteration scheme to construct the solution of (A).

THEOREM 3.6. *Assume that  $f$  is twice continuously differentiable,  $g_\xi(\xi, \eta)$  is continuous for  $\xi \geq 0$  and  $\eta \geq 0$ ,  $g_\eta(\xi, \eta) \geq 0$  for  $\xi \geq 0$ ,  $\eta \geq 0$ , and  $u_0(x) \geq 0$  on  $[0, 1]$ . Then there is a unique positive solution of (A) on some interval  $[0, T_0)$ , where  $T_0 = T(u_0) \leq \infty$ .*

PROOF: We set up a sequence  $\{u_k\}_{k=1}^\infty$  by the following procedure:

Let  $u_1 \equiv 0$  and  $u_{k+1}$  ( $k = 1, 2, \dots$ ) satisfy the problem

$$(C') \quad \begin{aligned} (u_{k+1})_t &= (u_{k+1})_{xx} + (f(u_{k+1}))_x & 0 < x < 1, \quad 0 < t < T, \\ (u_{k+1})(0, t) &= 0 & 0 < t < T, \\ (u_{k+1})_x(1, t) &= g(u_k(1, t), \bar{u}_k) & 0 < t < T, \\ (u_{k+1})(x, 0) &= u_0(x) & 0 \leq x \leq 1. \end{aligned}$$

The existence of a solution of  $(C')$  is guaranteed by results in [5]. Clearly,  $u_2(x, t) > 0$  for  $0 < x \leq 1, t > 0$  because  $(u_2)_x(1, t) = 0$ . Suppose that  $u_{k+1} \geq u_k$  on  $D_T$ ; then set  $\psi(x, t) = e^{\lambda x - 2\lambda^2 t}(u_{k+2} - u_{k+1})$  ( $\lambda > 0$ ), on  $D_T$   $\psi$  satisfies

$$\psi_t = \psi_{xx} + (-2\lambda + f'(u_{k+2}))\psi_x + [-\lambda^2 - \lambda f'(u_{k+2}) + f''(\theta_0)(u_{k+1})_x]\psi,$$

and at  $x = 1, 0 < t < T$

$$\psi_x \geq (g_\xi(\theta_1, \bar{u}_{k+1}) + \lambda)(u_{k+1} - u_k)e^{\lambda - 2\lambda^2 t},$$

where  $\theta_0$  is between  $u_{k+2}$  and  $u_{k+1}$ , and  $\theta_1$  is between  $u_{k+1}$  and  $u_k$ . We choose  $\lambda$  sufficiently large such that the coefficient of  $\psi$  is negative and  $(g_\xi + \lambda)$  positive. Then, since  $\psi$  cannot have a negative minimum inside and  $\psi_x(1, t) \geq 0$ , it follows that  $\psi \geq 0$ , i.e.,  $u_{k+2} \geq u_{k+1}$  on  $D_T$ . On the other hand, let  $g_M = \sup_{|\xi| \geq M} |g(\xi, \eta)|$ , since  $u_1 = 0$ , by induction and the preceding lemma, we can see that  $u_k \leq M$ ,  $k = 1, 2, \dots$  on  $D_{T_0}$ . Therefore,  $u(x, t) = \lim_{k \rightarrow \infty} u_k(x, t)$  exists.

Again, using Theorem 3 in [11], we can show that  $u$  is a solution of (A) and possesses the necessary interior regularity.

Uniqueness is guaranteed by the comparison principle.

Note that in constructing the positive solution, we did not use Theorem 3.1. For Burgers' equation, the four choices for  $g$  are therefore proper.

The same argument allows us to extend the solution to  $\bar{D}_{T+\delta}$  for some  $\delta > 0$ , if  $u$  remains bounded in  $\bar{D}_T$ . However, when the nonlinear terms and initial data

satisfy certain conditions (as shown below), there are no global solutions for (A) or (B).

**LEMMA 3.7.** *Let  $u(x, t)$  be a nonnegative solution of (A) with  $u_x(x, 0) \geq 0$ ,  $g(\xi, \eta) \geq 0$  for  $\xi, \eta \geq 0$ . Then  $u_x(x, t) \geq 0$  in  $D_T$ . Additionally, if  $f'(\xi) \geq 0$  for  $\xi \geq 0$  and  $g_\eta(\xi, \eta) \geq 0$  for  $\xi \geq 0, \eta \geq 0$ , then  $u(x, t) \geq v(x, t)$ , where  $v$  solves*

$$(D) \quad \begin{aligned} v_t &= v_{xx} & 0 < x < 1, \quad t > 0, \\ v(0, t) &= 0 & t > 0, \\ v_x(1, t) &= g(v, \bar{v}) & t > 0, \\ v(x, 0) &\leq u(x, 0) & 0 \leq x \leq 1. \end{aligned}$$

**PROOF:** Since  $u_x$  satisfies a linear equation with  $u_x$  nonnegative on the parabolic boundary, the first assertion follows. Then  $u$  is a supersolution of (D), and when the comparison theorem is applied, the second statement is valid.

Note that under the first assumption, we have  $u_x(x, t) \leq 0$  in  $D_T$  for (B). This fact will be used later.

**THEOREM 3.8.** *Suppose that  $f'(\xi) \geq 0$  for  $\xi \geq 0$ , and  $g(\xi, \eta) \geq 0$ ,  $g_\eta(\xi, \eta) \geq 0$  for  $\xi, \eta \geq 0$ , and*

$$(3.1) \quad (p+1)G(\xi, \eta) \leq \xi g(\xi, \eta) \quad \text{for } \xi, \eta \geq 0,$$

where  $G(\xi, \eta) = \int_0^\xi g(s, \eta) ds$ ,  $p > 1$ .

Furthermore, let  $v_0(x)$  satisfy

$$(3.2) \quad \begin{aligned} v_0(0) &= 0, & v_0(x) &\geq 0, & \text{and} & v_0''(x) + (f(v_0))' &\geq 0; \\ v_0'(1) &= g(v_0(1), \bar{v}_0), \end{aligned}$$

$$(3.3) \quad \frac{1}{2} \int_0^1 (v_0'(x))^2 dx < G(v_0(1), \bar{v}_0);$$

then, if  $u_0(x) \geq v_0(x)$  on  $[0, 1]$ , the solution of (A) blows up in finite time.

PROOF: By the lemma, it is sufficient to show that  $v(x, t)$  with  $v(x, 0) = v_0(x)$  blows up. Note that in this case, we have  $v_t \geq 0$  in  $D_T$ .

Using a function similar to that in [10], we define:

$$J(t) = \int_0^t \int_0^1 v^2(x, \eta) dx d\eta + (T^* - t) \int_0^1 v^2(x, 0) dx + \beta(t + \tau)^2,$$

where  $\tau, \beta, T^*$  are positive constants with  $T^* \geq T$ .

$$\begin{aligned} J'(t) &= \int_0^1 v^2(x, t) dx - \int_0^1 v^2(x, 0) dx + 2\beta(t + \tau) \\ &= 2 \int_0^t \int_0^1 v v_\eta dx d\eta + 2\beta(t + \tau) \\ &= -2 \int_0^t \int_0^1 v_x^2 dx d\eta + 2 \int_0^t v(1, \eta) g(v, \bar{v}) d\eta + 2\beta(t + \tau), \end{aligned}$$

and

$$J''(t) = -2 \int_0^1 v_x^2 dx + 2v(1, t) g(v, \bar{v}) + 2\beta.$$

Therefore,

$$\begin{aligned}
J''(t) &= -4 \int_0^t \int_0^1 v_{x\eta} v_x dx d\eta - 2 \int_0^1 v_0'^2 dx + 2v(1, t)g + 2\beta \\
&= 4 \int_0^t \int_0^1 v_\eta^2 dx d\eta - 4 \int_0^t v_\eta(1, \eta)g(v, \bar{v})d\eta - 2 \int_0^1 v_0'^2 dx + 2v(1, t)g + 2\beta \\
&= 4(\alpha + 1) \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right] + 2 \left[ -2 \int_0^t v_\eta(1, \eta)g(v, \bar{v})d\eta \right. \\
&\quad \left. - \int_0^1 v_0'^2 dx - 2\alpha \int_0^t \int_0^1 v_\eta v_{xx} dx d\eta + v(1, t)g(v, \bar{v}) - (2\alpha + 1)\beta \right] \\
&= 4(\alpha + 1) \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right] + 2 \left\{ \alpha \int_0^1 v_x^2 dx - (\alpha + 1) \int_0^1 v_0'^2 dx + v(1, t)g(v, \bar{v}) \right. \\
&\quad \left. - 2(\alpha + 1) \int_0^t \frac{\partial}{\partial \eta} \left( \int_{v_0(1)}^{v(1, \eta)} g(s, \bar{v}) ds \right) d\eta \right. \\
&\quad \left. + 2(\alpha + 1) \int_0^t \int_{v_0(1)}^{v(1, \eta)} g_{\bar{v}}(s, \bar{v}) \bar{v}_\eta ds d\eta - (2\alpha + 1)\beta \right\} \\
&\geq 4(\alpha + 1) \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right] + 2 \left\{ [v(1, t)g(v, \bar{v}) - 2(\alpha + 1)G(v, \bar{v})] \right. \\
&\quad \left. + 2(\alpha + 1) \left[ G(v_0(1), \bar{v}_0) - \frac{1}{2} \int_0^1 v_0'^2 dx - \frac{(2\alpha + 1)}{2(\alpha + 1)} \beta \right] \right\}.
\end{aligned}$$

Making use of (3.1) with  $\alpha = \frac{p-1}{2}$  and (3.3) with

$$\beta = \frac{2(\alpha + 1)}{(2\alpha + 1)} \left[ G(v_0(1), \bar{v}_0) - \frac{1}{2} \int_0^1 v_0'^2 dx \right],$$

we obtain

$$J''(t) \geq 4(\alpha + 1) \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right].$$

Hence,

$$\begin{aligned} JJ'' - (\alpha + 1)(J')^2 &\geq 4(\alpha + 1) \left\{ \left[ \int_0^t \int_0^1 v^2 dx d\eta + \beta(t + \tau)^2 \right] \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right] \right. \\ &\quad \left. - \left[ \int_0^t \int_0^1 vv_\eta dx d\eta + \beta(t + \tau) \right]^2 \right\} \geq 0. \end{aligned}$$

Consequently,  $(J^{-\alpha})'' \leq 0$ . Also,  $(J^{-\alpha}(0))' < 0$ . Therefore,  $J^{-\alpha}(t)$  has a positive root and thus the conclusion follows.

For example, consider Burgers' equation with  $g(\xi, \eta) = a\xi^p\eta^q$  ( $p > 1, q > 0$ ) and  $v_0(x) = Ax^r$ .

We discuss the case  $1 < p < 3$  only. For  $p \geq 3$ , since  $v_t \geq 0$ , if  $v_0(1) = A \geq 1$ , the corresponding solution  $v(x, t)$  is a supersolution of (A) with  $1 < p < 3$ , and hence by the comparison theorem,  $v(x, t)$  dominates the one for  $1 < p < 3$ .

For (3.2), we must require

$$(3.4) \quad Ar = \frac{aA^{p+q}}{(r+1)^q}, \quad A = \left[ \frac{r(r+1)^q}{a} \right]^{\frac{1}{p+q-1}}.$$

From (3.3),  $\frac{A^2 r^2}{4r-2} < \frac{aA^{p+q+1}}{(p+1)(r+1)^q}$ , combining with (3.4), we have  $(p+1)r^2 < r(4r-2)$ , it follows that  $r > \frac{2}{3-p}$ . Thus we choose  $r > \max \left\{ a, 2, \frac{2}{3-p} \right\}$ , which assures that  $A \geq 1$ .

For  $g(\xi, \eta) = a\xi^p\eta^q - \frac{1}{2}\varepsilon\xi^2$  ( $p \geq 2, q > 0$ ), we put  $u_0$  so large that  $au_0^q - \frac{1}{2}\varepsilon = b > 0$ . Then  $u$  dominates the solution of (D) with  $v_x(1, t) = bv^2(1, t)$ . Following the similar procedure as above, we can show that  $v$  blows up and so does  $u$ .

Next, we state a parallel result for problem (B) as follows:



**THEOREM 3.9.** Suppose that all hypotheses in Theorem 3.1 are satisfied and that  $g(\xi, \eta) \geq 0$  for  $\xi, \eta \geq 0$ . Define  $G(\xi, \eta) = \int_0^\xi g(s, \eta) ds$ ,  $F(\xi) = \int_0^\xi f(s) ds$ ; and assume that  $Q(\xi, \eta) \geq 0$  for some  $\alpha > 0$  and  $\xi \geq 0, \eta \geq k > 0$ , where

$$(3.5) \quad Q(\xi, \eta) = \xi g(\xi, \eta) - 2(\alpha + 1)F(\xi, \eta) + G(\xi) - \xi f(\xi).$$

Let  $u$  be a solution of (B) with  $u_0(x) \geq v_0(x)$  on  $[0, 1]$ . Here,  $v_0(x)$  satisfies

$$(3.6) \quad v_0(x) \geq 0, \quad v'_0(x) \leq 0, \quad v_0(1) = 0, \quad v''_0(x) + (f(v_0))' \geq 0,$$

$$\text{and } \int_0^1 v_0(x) dx \geq k;$$

$$(3.7) \quad -v'_0(0) = g(v_0(0), \bar{v}_0),$$

$$(3.8) \quad \frac{1}{2} \int_0^1 v'^2_0 dx < G(v_0(0), \bar{v}_0),$$

then,  $u$  will become unbounded in finite time.

**PROOF:** Let  $v(x, t)$  be the solution of (B) with  $v(x, 0) = v_0(x)$ , it is clear that  $v_t \geq 0$  and  $v_x \leq 0$  in  $D_T$ . We still define the same function  $J(t)$  as in Theorem 3.8, but

this time it leads to

$$\begin{aligned}
J''(t) &= -2v(0, t)f(v) + 2F(v) + 2v(0, t)g(v, \bar{v}) - 2 \int_0^1 v_x^2 dx + 2\beta \\
&= 2Q(v(0, t), \bar{v}) - 4(\alpha + 1) \int_0^t \int_0^1 v_x v_\eta f'(v) dx d\eta + 4(\alpha + 1) \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right] \\
&\quad + 2 \left[ -2 \int_0^t v_\eta(0, \eta)g(v, \bar{v})d\eta + G(v(0, t), \bar{v}) - \int_0^1 v_0'^2 dx \right. \\
&\quad \left. - 2\alpha \int_0^t \int_0^1 v_\eta v_{xx} dx d\eta - (2\alpha + 1)\beta \right] \\
&\geq 2Q(v(0, t), \bar{v}) + 4(\alpha + 1) \left[ \int_0^t \int_0^1 v_\eta^2 dx d\eta + \beta \right] \\
&\quad + 2(\alpha + 1) \left[ G(v_0(0), \bar{v}_0) - \frac{1}{2} \int_0^1 v_0'^2 dx - \frac{(2\alpha + 1)}{2(\alpha + 1)}\beta \right].
\end{aligned}$$

Then, combining (3.5), (3.6) and (3.8) yields the desired result.

As an application, take  $f(\xi) = \frac{1}{2}\varepsilon\xi^2$ ,  $g(\xi, \eta) = a\xi^p\eta^q$  ( $p \geq 2$ ,  $q > 0$ ) and  $v_0(x) = A(1-x)^r$  ( $A \geq 1$ ,  $r \geq 3$ ).

In view of the previous reason, we need to consider only the case  $p = 2$ .

Conditions (3.7) and (3.8) require that  $r > \frac{2}{3-p} = 2$  and  $A = \left[ \frac{r(r+1)^q}{a} \right]^{1/(q+1)}$ . Additionally, we have to check  $v_0''(x) + \varepsilon v_0(x)v_0'(x) \geq 0$ , which implies that  $(r-1) \geq \varepsilon A = \varepsilon \left[ \frac{r(r+1)^q}{a} \right]^{1/(q+1)}$ , that is,  $(r-1)^{q+1} \geq \varepsilon^{q+1} \frac{r(r+1)^q}{a}$ . Since  $r \geq 3$ , it follows that  $2(r-1) \geq r+1$ ; thus, it suffices to require  $(r-1)^{q+1} > \varepsilon^{q+1} \frac{2^{q+1}(r-1)^{q+1}}{a}$ . Consequently, we find

$$(3.9) \quad a > (2\varepsilon)^{q+1}.$$

Then let  $k = \int_0^1 v_0(x)dx = \frac{A}{r+1} = \left[ \frac{r}{a(r+1)} \right]^{1/(q+1)}$ . In order to check (3.5), we see

that the following should be valid:

$$a \left( 1 - \frac{2}{3}(\alpha + 1) \right) \xi^3 k^q - \frac{1}{3} \varepsilon \xi^3 \geq 0 \quad \text{for } \xi \geq 0,$$

which can be done if  $ak^q > \varepsilon$ , that is,  $a \left[ \frac{r}{a(r+1)} \right]^{q/(q+1)} > \varepsilon$ ,  $a > \varepsilon^{q+1} \left( 1 + \frac{1}{r} \right)^q$ , this is automatically satisfied if (3.9) holds.

For problem (B) with  $g(\xi, \eta) = a\xi^p \eta^q + \frac{1}{2} \varepsilon \xi^2$  and  $u_0 \geq v_0$ , the solution blows up because the solution of (B) with  $g(\xi, \eta) = a\xi^p \eta^q$  is a subsolution.

**REMARK 3.2:** It is worth mentioning that all results in this section can be extended to a multidimensional analog of (A):

$$\begin{aligned} (P_n) \quad & u_t = \Delta u + \nabla f(u) & x \in \Omega, \quad 0 < t < T, \\ & u(x, t) = 0 & x \in \sigma, \quad 0 < t < T, \\ & \frac{\partial u}{\partial n}(x, t) = g(u(x, t), \bar{u}) & x \in \Sigma, \quad 0 < t < T, \\ & u(x, 0) = u_0(x) & x \in \Omega. \end{aligned}$$

Here,  $f$  and  $g$  are as before,  $\Omega$  is a domain in  $\mathbb{R}^n$  ( $n \geq 2$ ),  $\text{cl}(\sigma \cup \Sigma) = \partial\Omega$ , and  $\sigma \cap \Sigma = \emptyset$ .

For instance, the comparison theorem can be shown in the same manner as that of Theorem 3.1; and because the proof of local existence in Theorem 3.6 does not depend on the Green's function associated with the operator  $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ , it can be used for  $(P_n)$  with one substitution  $e^{\lambda\rho - \mu t}w$  for  $e^{\lambda x - \mu t}w$ , where  $\rho$  satisfies  $\rho \geq 0$  in  $\Omega$ ,  $\frac{\partial \rho}{\partial n} \geq 1$  on  $\Sigma$ . Such  $\rho$  was given in Lemma 6 of [9].

#### 4. Stability and Instability Analysis

In this section, we shall study the long time behavior of global solutions. We use two different approaches to build up the criteria for stability and instability.

First, we state a lemma, whose proof is similar to that for Theorem 3.2A in [8] and hence is omitted.

**LEMMA 4.1.** *Let  $u(x, t)$  be a bounded monotone solution of (A) or (B), then  $u(x, t)$  exists for all time and tends to a stationary solution of (A) or (B).*

Now with somewhat weak assumptions, we can show one instability result:

**THEOREM 4.2A.** *Let  $f$  and  $g$  satisfy all hypotheses in Theorem 2.3A. If  $w(x)$  is the corresponding stationary solution of (A), then  $w(x)$  is unstable.*

**PROOF:** By Theorem 2.3A, we know that  $w(x)$  is the unique positive solution of  $(A_1)$ . Set  $v(x) = (1 - \delta_0)w(x)$  ( $0 < \delta_0 < 1$ ), we have

$$\begin{aligned} v_{xx} + f'(v)v_x &\leq (1 - \delta_0)[w_{xx} + f'((1 - \delta_0)w)w_x] \\ &\leq (1 - \delta_0)[w_{xx} + f'(w)w_x] \\ &\leq 0. \end{aligned}$$

Moreover, at  $x = 1$

$$\begin{aligned} v_x - g(v, \bar{v}) &= (1 - \delta_0)g(w, \bar{w}) - g((1 - \delta_0)w, (1 - \delta_0)\bar{w}) \\ &= (1 - \delta_0)w[g(w, \bar{w})/w - g((1 - \delta_0)w, (1 - \delta_0)\bar{w})/(1 - \delta_0)w] \\ &\geq 0. \end{aligned}$$

Hence,  $v(x)$  is a supersolution, and it follows that any solution  $u_\delta(x, t)$  of (A) with  $u_\delta(x, 0) = (1 - \delta)w(x)$  ( $\delta_0 < \delta < 1$ ) is bounded away from  $v(x)$  and monotonically increasing as  $\delta \rightarrow \delta_0^+$ . Therefore the solution  $u(x, t)$  with  $u(x, 0) = (1 - \delta_0)w(x)$  satisfies  $u(x, t) \leq v(x)$ , and  $u_t \leq 0$  in  $D_T$  as Corollary 3.3 is applied. The preceding lemma shows that  $u(x, t)$  goes to zero as  $t \rightarrow \infty$ , which implies that  $w(x)$  is unstable from below. Using  $u(x, t)$  with  $u(x, 0) = (1 + \delta_1)w(x)$ , we can show that  $w(x)$  is unstable from above.

In more or less the same manner, we prove a counterpart for  $(B_1)$ .

**THEOREM 4.2B.** *Let all conditions in Theorem 2.3B be satisfied, and let  $w(x)$  be the unique stationary solution of  $(B_1)$ . Then  $w(x)$  is stable.*

**PROOF:** Let  $v(x) = (1 + \delta_0)w(x)$  ( $\delta_0 > 0$ ). For any solution  $u(x, t)$  of (B) with  $u(x, 0) = v(x)$ , following the same argument as in Theorem 4.2A, we can show that  $w(x) \leq u(x, t) \leq (1 + \delta_0)w(x)$  and  $u_t(x, t) < 0$  in  $D_T$ . Hence,  $u(x, t)$  approaches  $w(x)$  as  $t \rightarrow \infty$ , which means that  $w(x)$  is stable from above. A similar argument can show that  $w(x)$  is stable from below.

In order to obtain a more complete result for stability and instability, we have to assume a few more conditions on the nonlinear terms  $f$  and  $g$ . This time, we replace  $f$  by  $\varepsilon f$  ( $\varepsilon \geq 0$ ); thus the stationary solution  $w(x) = w(x, \varepsilon)$  can be treated as a function depending on the parameter  $\varepsilon$ .

**THEOREM 4.3A.** *Suppose that  $f$  is twice continuously differentiable,  $f'(\xi) > 0$  if  $\xi > 0$ ,  $g_\xi(\xi, \eta)$  is continuous for  $\xi \geq 0$ ,  $\eta \geq 0$  and  $g_\eta(\xi, \eta) \geq 0$  for  $\xi, \eta \geq 0$ . Let  $w(x, \varepsilon)$  be a continuously differentiable positive solution of  $(A_1)$  on some  $\varepsilon$  interval  $[a, b]$  and let  $w_1(\varepsilon) = w(1, \varepsilon)$ . Then, if  $w'_1(\varepsilon) \geq 0$  on  $[a, b]$ , the solutions are stable, whereas they are unstable if  $w'_1(\varepsilon) < 0$ .*

**PROOF:** For the case  $w'_1(\varepsilon) \geq 0$ , set  $v(x) = \frac{\partial w}{\partial \varepsilon}(x, \varepsilon)$ , we claim that  $v(x) > 0$  on  $(0, 1)$  for  $a \leq \varepsilon \leq b$ . To see this, suppose that  $x_0$  is a point in  $(0, 1)$  such that  $v(x_0) \leq 0$ ,  $v'(x_0) = 0$ . From the conservation law, we have

$$v' + \varepsilon f'(w)v = f(w(x_0)) - f(w(x)) \quad 0 < x < 1$$

and

$$v(0) = 0, \quad v(1) \geq 0.$$

By using the formula for linear equation, we find that

$$v(x_0) = e^{\int_0^1 \epsilon f'(w(s)) ds} \left( v(1) + \int_{x_0}^1 (f(w(\tau)) - f(w(x_0))) d\tau \right) e^{\int_1^x \epsilon f'(w(s)) ds} > 0,$$

which is a contradiction.

From the above assertion, it follows that  $w(x, \epsilon_1) < w(x, \epsilon_2)$  on  $(0, 1)$  for  $a \leq \epsilon_1 < \epsilon_2 \leq b$ .

Let  $u(x, t, \epsilon_1)$  be a solution of (A) with  $u_0(x, \epsilon_1) = w(x, \epsilon_2)$ . Then, on  $(0, 1)$ , we have

$$\begin{aligned} u_0'' + \epsilon_1 f'(u_0) u_0' &= w_{xx}(x, \epsilon_2) + \epsilon_1 f'(w(x, \epsilon_2)) w_x(x, \epsilon_2) \\ &< w_{xx}(x, \epsilon_2) + \epsilon_2 f'(w(x, \epsilon_2)) w_x(x, \epsilon_2) \\ &= 0. \end{aligned}$$

Hence, recalling Corollary 3.4, we have  $u_t < 0$  in  $D_T$ .

From the comparison theorem and monotonicity of  $u$ , we also have, on  $(0, 1)$ ,

$$w(x, \epsilon_1) < u(x, t, \epsilon_1) \leq w(x, \epsilon_2).$$

By Lemma 4.1,  $\phi(x, \epsilon_1) = \lim_{t \rightarrow \infty} u(x, t, \epsilon_1)$  exists, and  $w(x, \epsilon_1) \leq \phi(x, \epsilon_1) \leq w(x, \epsilon_2)$ . Letting  $\epsilon_2 \rightarrow \epsilon_1^+$  yields  $\phi(x, \epsilon_1) \equiv w(x, \epsilon_1)$ , which shows that  $w(x, \epsilon_1)$  is stable from above. With  $\epsilon_1 > \epsilon_2$ , in a similar manner, we can also prove that  $w(x, \epsilon_1)$  is stable from below.

If  $w_1'(\epsilon) < 0$ , we know that  $w(x, \epsilon_2) < w(x, \epsilon_1)$  in a subinterval  $[x_1, 1]$  for  $a \leq \epsilon_1 < \epsilon_2 \leq b$ . Now let  $u(x, t, \epsilon_2)$  be a solution of (A) with  $u_0(x, \epsilon_2) = w(x, \epsilon_1)$ . Then, on  $(0, 1)$ , we find that

$$\begin{aligned} u_0'' + \epsilon_2 f'(u_0) u_0' &= w_{xx}(x, \epsilon_1) + \epsilon_2 f'(w(x, \epsilon_1)) w_x(x, \epsilon_1) \\ &> w_{xx}(x, \epsilon_1) + \epsilon_1 f'(w(x, \epsilon_1)) w_x(x, \epsilon_1) \\ &= 0. \end{aligned}$$

Therefore,  $u_t > 0$  in  $D_T$ . Consequently,  $u(x, t, \varepsilon_2)$  is increasing in  $t$ , which indicates that  $w(x, \varepsilon_2)$  is unstable from above. Similarly, with  $\varepsilon_1 > \varepsilon_2$ , we can show that  $w(x, \varepsilon_2)$  is unstable from below.

As a parallel result, for problem (B), we obtain the following:

**THEOREM 4.3B.** *Assume that  $f$  is twice continuously differentiable,  $f'(\xi) > 0$  and  $f''(\xi) \geq 0$  for  $\xi > 0$ ,  $g_\xi(\xi, \eta)$  is continuous for  $\xi \geq 0$ ,  $\eta \geq 0$  and  $g_\eta(\xi, \eta) \geq 0$  for  $\xi, \eta \geq 0$ . Let  $w(x, \varepsilon)$  be a  $C^1$  (in  $\varepsilon$ ) positive stationary solution for  $a \leq \varepsilon \leq b$ , and let  $w_0(\varepsilon) = w(0, \varepsilon)$ . Then if  $w'_0(\varepsilon) \leq 0$ , the solutions are stable; whereas they are unstable if  $w'_0(\varepsilon) > 0$  on  $[a, b]$ .*

The proof is actually the same as that for Theorem 3.6B in [8], and is therefore omitted.

In particular, with the help of those theorems, we are able to establish the crucial criteria and to give the bifurcation diagrams for Burgers' equation with  $f(\xi) = \frac{1}{2}\varepsilon\xi^2$  and  $g(\xi, \eta) = a\xi^p\eta^q$  or  $g(\xi, \eta) = a\xi^p\eta^q \pm \frac{1}{2}\varepsilon\xi^2$ .

For problem (A<sub>1</sub>) with  $g(\xi, \eta) = a\xi^p\eta^q$ , combining (2.3) and (2.4) yields

$$(4.1) \quad \varepsilon = w_1^{-1}(\varepsilon)\gamma^{-1} \ln \left( \frac{\gamma+1}{\gamma-1} \right).$$

By replacing  $\varepsilon$  in (2.3) with (4.1), we obtain

$$(4.2) \quad [w_1(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left( \frac{1}{2a} \right)^{\frac{1}{1+q}} (\gamma^2 - 1)^{\frac{1}{1+q}} \gamma^{-1} \ln \left( \frac{\gamma+1}{\gamma-1} \right) \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^{-\frac{q}{1+q}}.$$

First, we discuss the case  $p+q \neq 1$ .

(4.2) can be written in the form of

$$[w_1(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \Phi(\gamma),$$

where  $\Phi(\gamma)$  is the same as in (2.8) with  $\lambda = \frac{1}{1+q}$ ,  $\mu = \frac{q}{1+q}$ . Hence,

$$(4.3) \quad \left( \frac{p+q-1}{1+q} \right) [w_1(\varepsilon)]^{\frac{p-2}{1+q}} w'_1(\varepsilon) = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \Phi'(\gamma) \gamma'(\varepsilon).$$

Since  $\Phi'(\gamma) > 0$  on  $(1, \infty)$ , if  $p + q > 1$ ,  $w'_1(\varepsilon)$  has the same sign as  $\gamma'(\varepsilon)$ ; whereas it has the opposite sign if  $p + q < 1$ .

If  $p = 2$ , from (2.3), we have

$$\varepsilon^{1+q} = \varphi(\gamma) = 2a(\gamma^2 - 1)^{-1} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^q.$$

It is clear that  $\gamma'(\varepsilon) < 0$  because  $\varphi'(\gamma) < 0$ . Hence, when  $p = 2$ ,  $w(x, \varepsilon)$  is an unstable branch. For  $p \neq 2$ , recalling the equation  $\Phi(\gamma) = \delta$  in (2.7), we find that

$$\Phi'(\gamma)\gamma'(\varepsilon) = \left( \frac{p+q-1}{p-2} \right) 2^{\frac{1-p}{p-2}} a^{\frac{1}{p-2}} \varepsilon^{\frac{1+q}{p-2}}.$$

If  $p > 2$ , we have that  $\gamma'(\varepsilon) < 0$  because  $\Phi'(\gamma) < 0$ . The same holds for  $1 \leq p < 2$  and  $0 < p < 1$  ( $p + q > 1$ ) because  $\Phi'(\gamma) > 0$  and  $(p+q-1)/(p-2) < 0$ .

Next, consider that  $p + q < 1$ . We observe that  $\Phi'(\gamma)$  is first positive and then negative, it follows that there are two branches, one satisfying  $\gamma'(\varepsilon) > 0$  and another satisfying  $\gamma'(\varepsilon) < 0$ .

Finally, for  $p + q = 1$ , because  $\Phi'(\gamma) > 0$  on  $(1, \infty)$ , it is clear from (4.3) that  $\gamma'(\varepsilon) = 0$ . Using (4.1), we can see that  $w'_1(\varepsilon) < 0$ .

If  $p + q > 1$ , we can obtain the same result by means of Theorem 4.2A.

Therefore, if  $p \geq 1$  or if  $0 < p < 1$  but  $p + q \geq 1$ ,  $w(x, \varepsilon)$  is unstable; whereas there are two branches of solutions for  $p + q < 1$ —one unstable, the other stable.

Then, for  $(B_1)$  with  $g(\xi, \eta) = a\xi^p\eta^q$ , we first observe that

$$(4.4) \quad \varepsilon = 2w_0^{-1}(\varepsilon)\gamma^{-1} \tan^{-1} \frac{1}{\gamma}.$$

Substituting (4.4) into (2.9), we find that

$$(4.5) \quad [w_0(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \tan^{-1} \frac{1}{\gamma} \gamma^{-1} (\gamma^2 + 1)^{\frac{1}{1+q}} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\frac{q}{1+q}}.$$

If  $p + q \neq 1$ , (4.5) can be rewritten as

$$[w_0(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \Psi(\gamma).$$



Here,  $\Psi(\gamma)$  is defined as in (2.14), with  $\lambda = \frac{1}{1+q}$ ,  $\mu = \frac{q}{1+q}$ . Similarly,

$$(4.6) \quad \left( \frac{p+q-1}{1+q} \right) [w_0(\varepsilon)]^{\frac{p-2}{1+q}} w'_0(\varepsilon) = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \Psi'(\gamma) \gamma'(\varepsilon).$$

Because  $\Psi'(\gamma) < 0$  for  $\gamma > 0$ , then if  $p+q > 1$ ,  $w'_0(\varepsilon)$  has the opposite sign to  $\gamma'(\varepsilon)$ ; but if  $p+q < 1$ , it has the same sign.

If  $p = 2$ , by (2.9), we obtain  $\varepsilon^{1+q} = \psi(\gamma) = 2a(\gamma^2 + 1)^{-1} \left[ \ln \left( \frac{\gamma^2+2}{\gamma^2+1} \right) \right]^q$  and find  $\gamma'(\varepsilon) < 0$ ; hence,  $w'_0(\varepsilon) > 0$ .

For  $p > 2$ , the fact that  $\Psi'(\gamma) < 0$  leads to  $\gamma'(\varepsilon) < 0$ . But for  $1 \leq p < 2$  or  $0 < p < 1$  ( $p+q > 1$ ), we can see that  $\Psi'(\gamma)$  changes sign once from negative to positive. Therefore, there exist two branches:  $\gamma'(\varepsilon) > 0$  for one, and  $\gamma'(\varepsilon) < 0$  for the other.

Then, for the case  $p+q < 1$ , the result  $\gamma'(\varepsilon) < 0$  follows from  $\Psi'(\gamma) < 0$ .

At last, we discuss  $p+q = 1$ . Obviously,  $\gamma'(\varepsilon) = 0$ . Noticing (4.4), we see that  $w'_0(\varepsilon) < 0$ .

In summary, for  $p \geq 2$ ,  $w(x, \varepsilon)$  is unstable; whereas it is stable for  $p+q \leq 1$ . For  $1 \leq p < 2$  or  $0 < p < 1$  ( $p+q > 1$ ), there are two branches—one stable but the other unstable.

Now, we briefly discuss  $w_1(\varepsilon)$  and  $w_0(\varepsilon)$  graphically and draw the corresponding diagrams.

For  $(A_1)$ , we first attempt to study the relation between  $\varepsilon$  and  $\gamma$  based on (2.7), and then describe the behavior of  $w_1(\varepsilon)$ .

If  $1 \leq p \neq 2$ , or  $0 < p < 1$  but  $p+q > 1$ ,  $\varepsilon \rightarrow 0$  as  $\gamma \rightarrow \infty$  and  $\varepsilon \rightarrow +\infty$  as  $\gamma \rightarrow 1^+$ ; thus,  $w_1(0) = \left( \frac{2^q}{a} \right)^{\frac{1}{p+q-1}}$  and  $w_1(+\infty) = +\infty$ . The same is true for  $p = 2$  because  $\varepsilon^{1+q} = \varphi(\gamma)$ .

For  $p+q = 1$ , there is a unique solution  $w(x, \varepsilon)$  if  $a < 2^{1-p}$ . Since  $\varepsilon$  is independent of  $\gamma$ , by (4.1), we have that  $w_1(0) \rightarrow \infty$  and  $w_1(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow +\infty$ .

For  $p+q < 1$ ,  $\varepsilon$  ranges over a finite interval  $[0, \varepsilon]$  because  $0 \leq \Phi(\gamma) \leq M$ . Then

the stable branch of  $w(x, \varepsilon)$  increases from  $(\frac{2^q}{a})^{\frac{1}{p+q-1}}$  to  $w_1(\varepsilon_1)$  and the unstable one grows from  $w_1(\varepsilon_1)$  to infinity as  $\varepsilon \rightarrow 0^+$ .

Graphs are shown in Figs. 3.1 and 3.2.

Then, for (B<sub>1</sub>), using (2.13) and (4.5), we can carry out a similar discussion.

For  $p > 2$  and  $p + q < 1$ ,  $\varepsilon \rightarrow 0$  if  $\gamma \rightarrow \infty$  and  $\varepsilon \rightarrow +\infty$  as  $\gamma \rightarrow 0^+$ . Therefore,  $w_0(0) = (\frac{2^q}{a})^{\frac{1}{p+q-1}}$  for both cases. However,  $\lim_{\varepsilon \rightarrow +\infty} w_0(\varepsilon) = +\infty$  for  $p > 2$ ; whereas  $\lim_{\varepsilon \rightarrow +\infty} w_0(\varepsilon) = 0$  if  $p + q < 1$ . The case  $p = 2$  needs to be treated separately because now  $\varepsilon$  is bounded by  $\varepsilon_2 = (2a)^{\frac{1}{1+q}}(\ln 2)^{\frac{1}{1+q}}$ , and  $\varepsilon^{1+q} = \psi(\gamma)$  implies that  $\varepsilon \rightarrow \varepsilon_2$  as  $\gamma \rightarrow 0^+$ . Thus,  $w_0(0) = (\frac{2^q}{a})^{\frac{1}{p+q-1}}$  and  $\lim_{\varepsilon \rightarrow \varepsilon_2^-} w_0(\varepsilon) = +\infty$ .

For  $1 \leq p < 2$  and  $0 < p < 1$  ( $p + q > 1$ ),  $\varepsilon$  runs from 0 to  $\varepsilon_0 < \infty$ , since  $\Psi \geq m$  on  $(0, \infty)$ . For this reason, the unstable branch of  $w(x, \varepsilon)$  increases from  $(\frac{2^q}{a})^{\frac{1}{p+q-1}}$  to  $w_0(\varepsilon_0)$ ; whereas the stable one decreases from infinity to  $w_0(\varepsilon_0)$ .

Finally, for  $p + q = 1$ , note that  $\gamma'(\varepsilon) = 0$  and (4.4), so we find that  $\lim_{\varepsilon \rightarrow 0^+} w_0(\varepsilon) = +\infty$  and  $\lim_{\varepsilon \rightarrow +\infty} w_0(\varepsilon) = 0$  if  $2^{1-p} < a$ .

The solution diagrams are given in Figs. 3.3 and 3.4.

Using a similar argument, we can show the corresponding diagrams for problem (A<sub>1</sub>) with  $g(\xi, \eta) = a\xi^p\eta^q - \frac{1}{2}\varepsilon\xi^2$ , and for problem (B<sub>1</sub>) with  $g(\xi, \eta) = a\xi^p\eta^q + \frac{1}{2}\varepsilon\xi^2$ .

For (A<sub>1</sub>), we can find the same equation for  $\varepsilon$  as (4.1)

$$(4.7) \quad \varepsilon = w_1^{-1}(\varepsilon)\gamma^{-1} \ln \left( \frac{\gamma+1}{\gamma-1} \right).$$

By replacing  $\varepsilon$  in (2.15) with (4.7), we obtain

$$(4.8) \quad [w_1(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left( \frac{1}{2a} \right)^{\frac{1}{1+q}} \gamma^{\frac{1-q}{1+q}} \ln \left( \frac{\gamma+1}{\gamma-1} \right) \left[ \ln \left( \frac{\gamma^2}{\gamma^2-1} \right) \right]^{-\frac{q}{1+q}}.$$

First, we discuss the case  $p + q \neq 1$ .

(4.8) can be written in the form of

$$[w_1(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \Phi(\gamma),$$

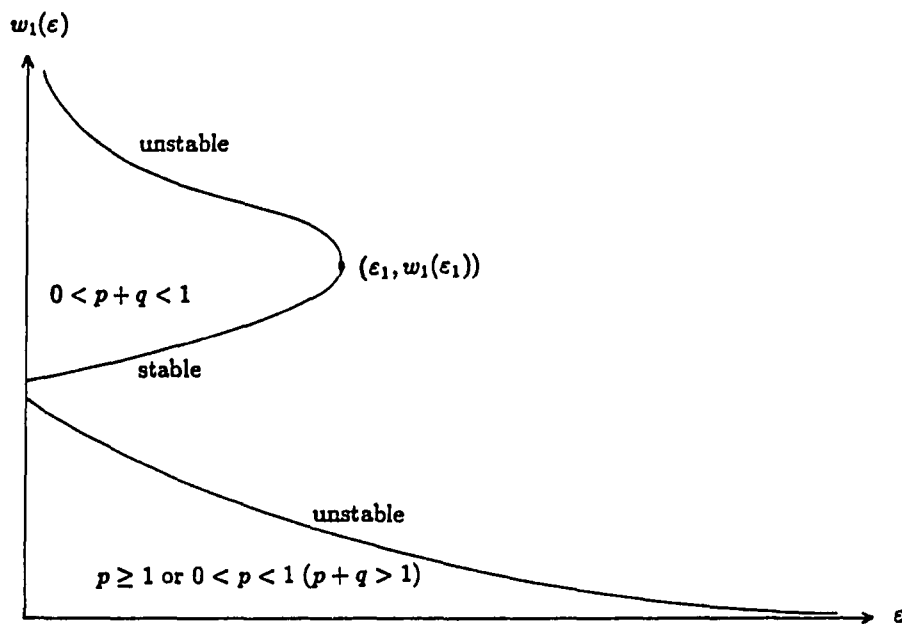


Figure 3.1:  $w_1(\varepsilon)$  for  $g(\xi, \eta) = a\xi^p\eta^q$ ,  $a \geq 2^{1-p}$  and  $w_1(0) = (2^q/a)^{\frac{1}{p+q-1}}$ .

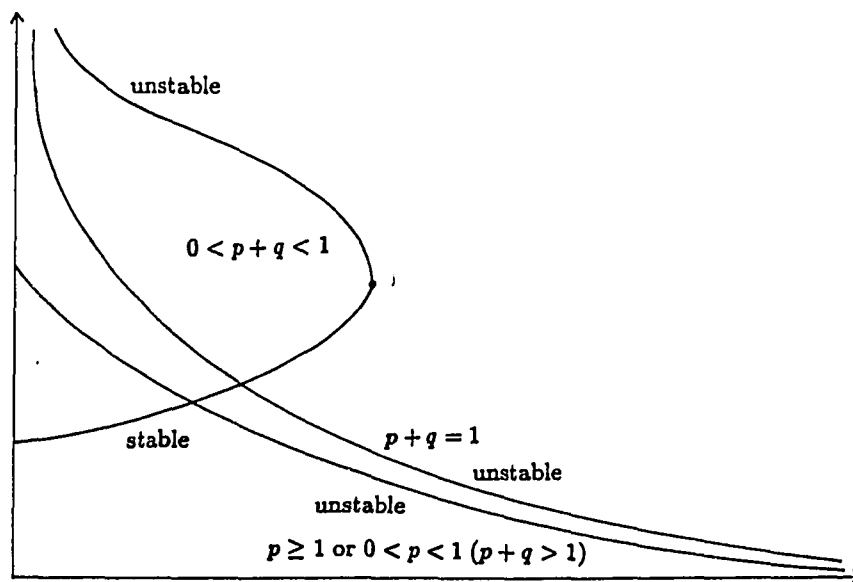


Figure 3.2: Same as above except  $0 < a < 2^{1-p}$ .

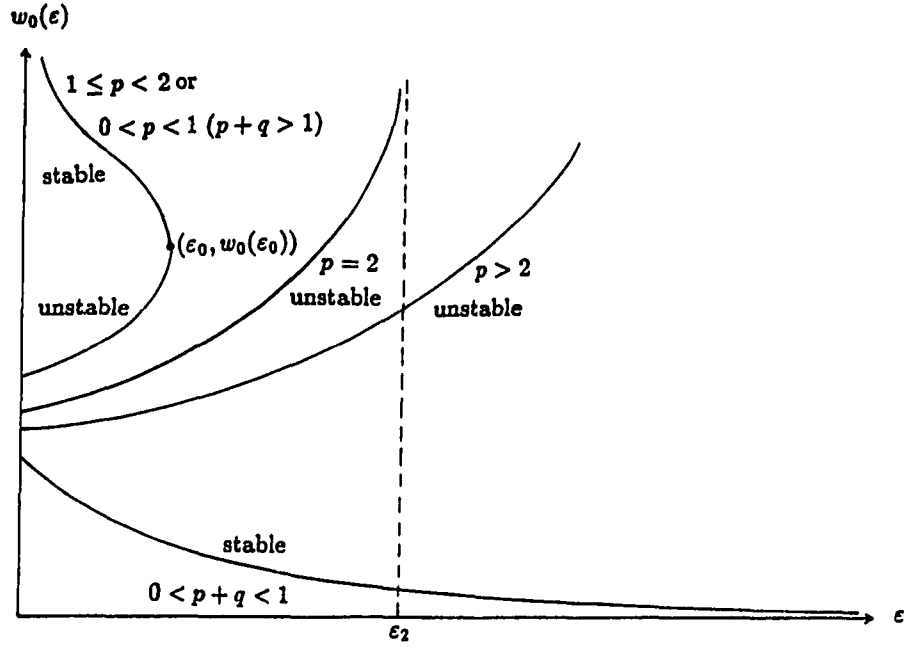


Figure 3.3:  $w_0(\epsilon)$  for  $g(\xi, \eta) = a\xi^p\eta^q$ ,  $0 < a \leq 2^{1-p}$  and  $w_0(0) = (2^q/a)^{\frac{1}{p+q-1}}$ .

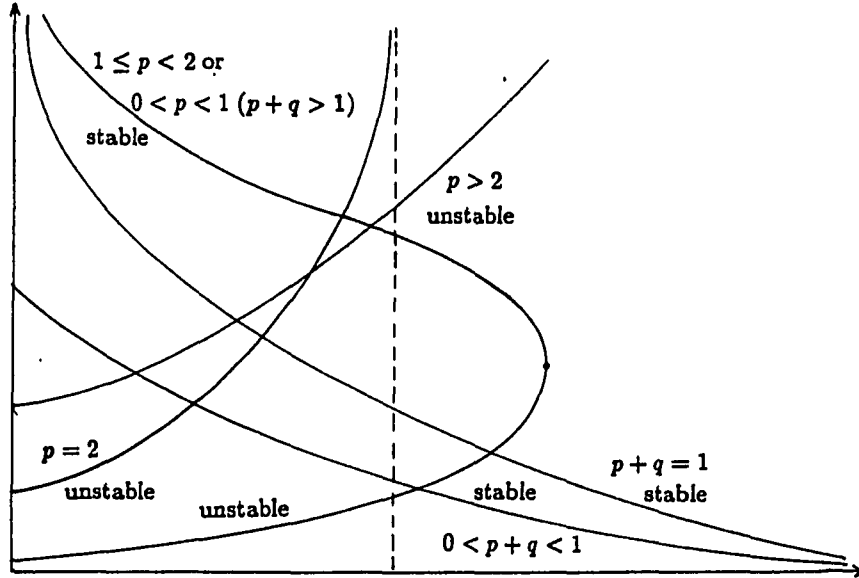


Figure 3.4: Same as above except  $a > 2^{1-p}$ .

where  $\Phi(\gamma)$  is the same as in (2.17) with  $\lambda = \frac{1}{1+q}$ ,  $\mu = \frac{q}{1+q}$ . Hence,

$$(4.9) \quad \left( \frac{p+q-1}{1+q} \right) [w_1(\varepsilon)]^{\frac{p-2}{1+q}} w_1'(\varepsilon) = \left( \frac{2^q}{a} \right)^{\frac{1}{1+q}} \Phi'(\gamma) \gamma'(\varepsilon).$$

Since  $\Phi'(\gamma) < 0$  on  $(1, \infty)$ , if  $p+q < 1$ ,  $w_1'(\varepsilon)$  has the same sign as  $\gamma'(\varepsilon)$ ; whereas it has the opposite sign if  $p+q > 1$ .

If  $p = 2$ , from (2.15), we have

$$\varepsilon^{1+q} = \varphi(\gamma) = 2a\gamma^{-2} \left[ \ln \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \right]^q.$$

It is clear that  $\gamma'(\varepsilon) < 0$  because  $\varphi'(\gamma) < 0$ .

For  $p \neq 2$ , recalling the equation  $\Phi(\gamma) = \delta$  in (2.16), we find that

$$\Phi'(\gamma) \gamma'(\varepsilon) = \left( \frac{p+q-1}{p-2} \right) 2^{\frac{1-p}{p-2}} a^{\frac{1}{p-2}} \varepsilon^{\frac{1+q}{p-2}}.$$

If  $p > 2$ , we have that  $\gamma'(\varepsilon) < 0$  because  $\Phi'(\gamma) < 0$ . The same holds for  $p < 2$  ( $p+q \geq 2$ ) because  $\Phi'(\gamma) > 0$  and  $(p+q-1)/(p-2) < 0$ , and for  $p+q < 1$  since  $\Phi'(\gamma) < 0$ .

Next, consider that  $1 < p+q < 2$ . We observe that  $\Phi'(\gamma)$  is first negative and then positive, it follows that there are two branches, one satisfying  $\gamma'(\varepsilon) > 0$  and another satisfying  $\gamma'(\varepsilon) < 0$ .

Finally, for  $p+q = 1$ , because  $\Phi'(\gamma) < 0$  on  $(1, \infty)$ , it is clear from (4.9) that  $\gamma'(\varepsilon) = 0$ . Using (4.7), we can see that  $w_1'(\varepsilon) < 0$ .

Therefore, if  $p+q \geq 2$ ,  $w(x, \varepsilon)$  is stable; if  $p+q \leq 1$ ,  $w(x, \varepsilon)$  is unstable; whereas there are two branches of solutions for  $1 < p+q < 2$  — one unstable, the other stable.

Graphs are shown in Figs. 4.1 and 4.2.

Then, for  $(B_1)$ , we first observe that

$$(4.10) \quad \varepsilon = 2w_0^{-1}(\varepsilon) \gamma^{-1} \tan^{-1} \frac{1}{\gamma}.$$

Substituting (4.10) into (2.18), we find that

$$[w_0(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left(\frac{2^q}{a}\right)^{\frac{1}{1+q}} \tan^{-1} \frac{1}{\gamma} \gamma^{\frac{1-q}{1+q}} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^{-\frac{1}{1+q}}.$$

If  $p + q \neq 1$ , (4.11) can be rewritten as

$$[w_0(\varepsilon)]^{\frac{p+q-1}{1+q}} = \left(\frac{2^q}{a}\right)^{\frac{1}{1+q}} \Psi(\gamma).$$

Here,  $\Phi(\gamma)$  is defined as in (2.20), with  $\lambda = \frac{1}{1+q}$ ,  $\mu = \frac{q}{1+q}$ . Similarly,

$$(4.12) \quad \left(\frac{p+q-1}{1+q}\right) [w_0(\varepsilon)]^{\frac{p-2}{1+q}} w'_0(\varepsilon) = \left(\frac{2^q}{a}\right)^{\frac{1}{1+q}} \Psi'(\gamma) \gamma'(\varepsilon).$$

Suppose  $\Psi'(\gamma) > 0$  for  $\gamma > 0$ , an assumption supported by numerical computation, then if  $p + q < 1$ ,  $w'_0(\varepsilon)$  has the opposite sign to  $\gamma'(\varepsilon)$ ; but if  $p + q > 1$ , it has the same sign.

If  $p = 2$ , by (2.18), we obtain  $\varepsilon^{1+q} = \psi(\gamma) = 2a\gamma^{-2} \left[ \ln \left( \frac{\gamma^2 + 2}{\gamma^2 + 1} \right) \right]^q$  and find  $\gamma'(\varepsilon) < 0$ .

For  $p > 2$ , the fact that  $\Psi'(\gamma) < 0$  leads to  $\gamma'(\varepsilon) < 0$ . The same holds for  $1 \leq p < 2$  and for  $p < 1$  ( $p + q > 1$ ) since  $\Psi'(\gamma) > 0$ .

Then for the case  $p + q < 1$ , we can see that  $\Psi'(\gamma)$  changes sign once from positive to negative. Therefore, there exist two branches:  $\gamma'(\varepsilon) > 0$  for one, and  $\gamma'(\varepsilon) < 0$  for the other.

At last, we discuss  $p + q = 1$ . Obviously,  $\gamma'(\varepsilon) = 0$ . Noticing (4.10), we see that  $w'_0(\varepsilon) < 0$ .

In summary, for  $p \geq 1$  and  $0 < p < 1$  ( $p + q \geq 1$ ),  $w(x, \varepsilon)$  is stable; whereas for  $0 < p + q < 1$ , there are two branches — one stable but the other unstable.

The solution diagrams are given in Figs. 4.3 and 4.4.

At the end of this section, we investigate stability of stationary solutions of Burgers' equation with  $g(\xi, \eta) = a\xi^p\eta^q$  or  $g(\xi, \eta) = a\xi^p\eta^q \pm \frac{1}{2}\varepsilon\xi^2$  ( $a < 0$ ). The positive solutions of (A) or (B) can be obtained by a contraction method, as in [8].

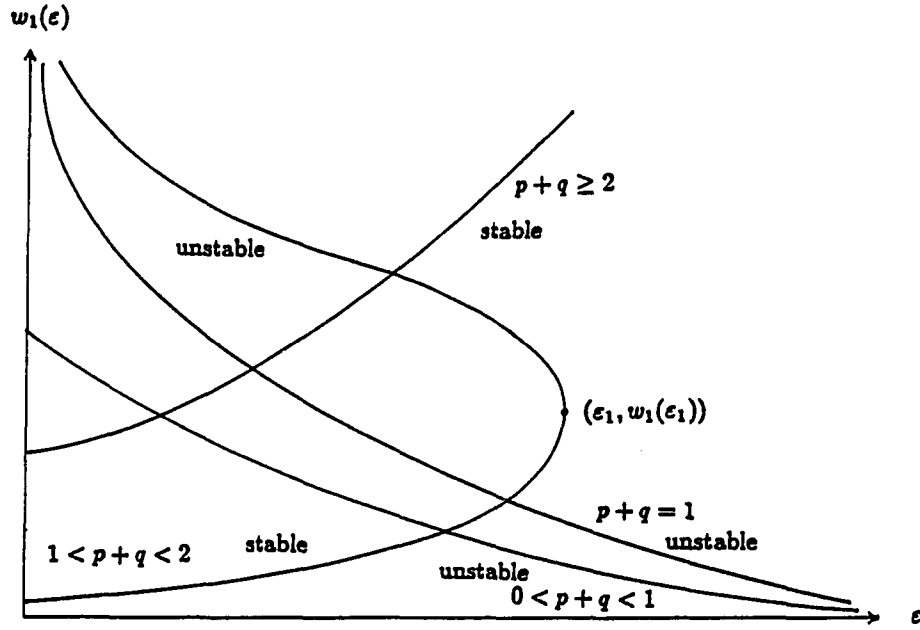


Figure 4.1:  $w_1(\varepsilon)$  for  $g(\xi, \eta) = a\xi^p\eta^q - \frac{1}{2}\varepsilon\xi^2$ ,  $a > 2^{1-p}$  and  $w_1(0) = (2^q/a)^{\frac{1}{p+q-1}}$ .

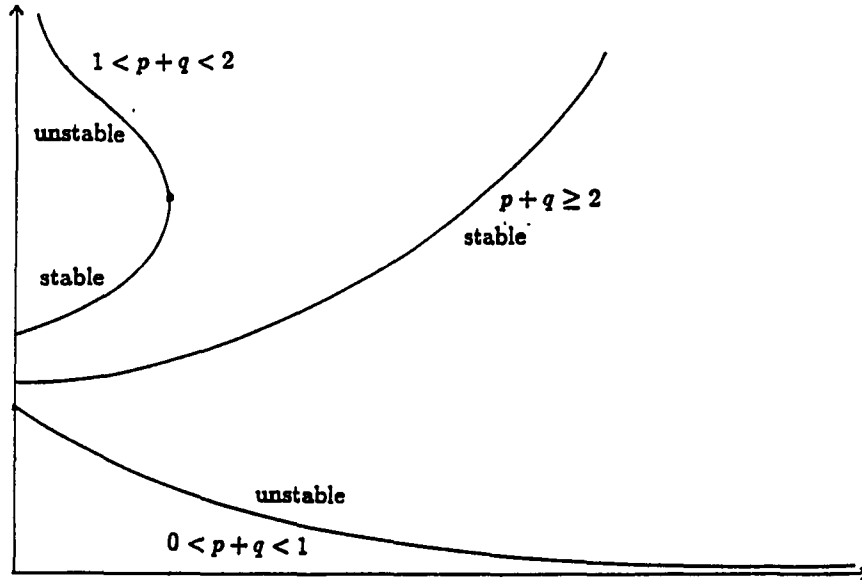


Figure 4.2: Same as above except  $0 < a \leq 2^{1-p}$ .

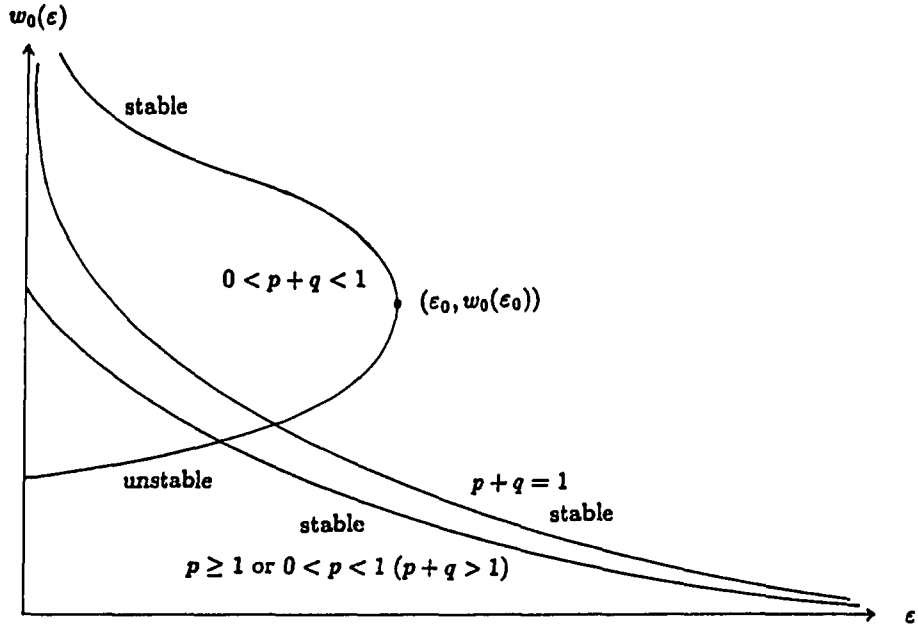


Figure 4.3:  $w_0(\epsilon)$  for  $g(\xi, \eta) = a\xi^p\eta^q + \frac{1}{2}\epsilon\xi^2$ ,  $0 < a < 2^{1-p}$  and  $w_0(0) = (2^q/a)^{\frac{1}{p+q-1}}$ .

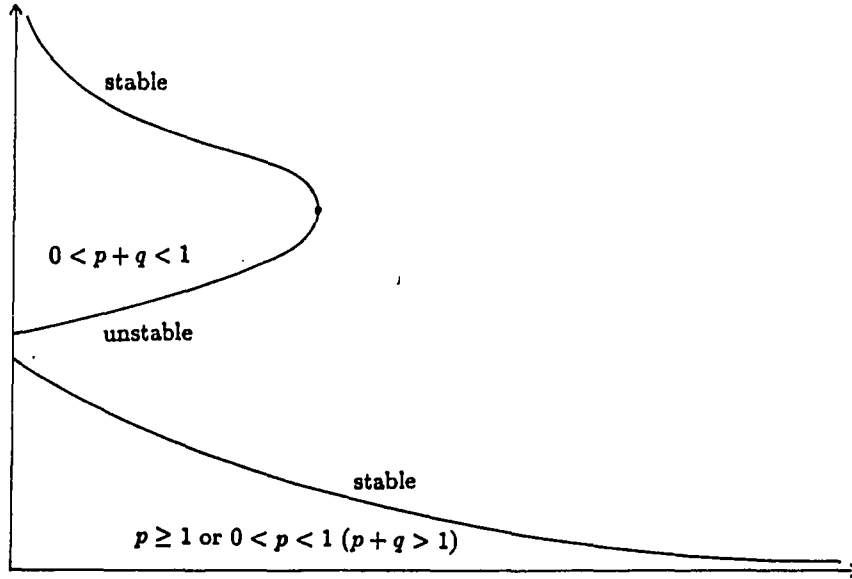


Figure 4.4: Same as above except  $a \geq 2^{1-p}$ .



First, consider problem (A) with  $g(\xi, \eta) = \xi^2$ . For convenience, we denote its solution and its stationary solution by  $u_A(x, t)$  and  $w_A(x)$ , respectively. By the comparison principle, it follows that the solution  $u(x, t)$  of (A) with  $g(\xi, \eta) = a\xi^p\eta^q$  or  $g(\xi, \eta) = a\xi^p\eta^q - \frac{1}{2}\varepsilon u^2$  satisfies: if  $0 \leq u(x, 0) \leq u_A(x, t)$ , then  $0 \leq u(x, t) \leq u_A(x, t)$ . Choosing  $u_A(x, 0) = (1 - \delta)w_A(x)$  and following a process similar to that in proving Theorem 4.2A, we can see that  $u_A(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ , which implies that the null solution of  $(A_1)$  is stable from above.

Next, consider problem (B) with  $g(\xi, \eta) = \frac{1}{2}\varepsilon\xi^2$ , whose solution and stationary solution may be denoted without confusion by  $u_B(x, t, \varepsilon)$  and  $w_B(x, \varepsilon)$ , respectively. Again, the comparison theorem shows that the solution of (B) with  $g(\xi, \eta) = a\xi^p\eta^q$  or  $g(\xi, \eta) = a\xi^p\eta^q + \frac{1}{2}\varepsilon\xi^2$  is bounded by  $u_B(x, t, \varepsilon)$  if  $u(x, 0) \leq u_B(x, 0, \varepsilon)$ .

Set  $u_B(x, 0, \varepsilon) = w_B(x, \sigma)$  ( $\sigma < \varepsilon$ ) to find

$$\begin{aligned} u_{Bt}(x, 0, \varepsilon) &= w_B'' + \varepsilon w_B w_B' \\ &= w_B'' + \sigma w_B w_B' + (\varepsilon - \sigma) w_B w_B' \\ &< 0, \end{aligned}$$

since  $w_B' < 0$ . From this, we conclude as before, that  $u_{Bt}(x, t) \leq 0$  in  $D_T$ . Note that  $w_B(x, \sigma) < w_B(x, \varepsilon)$  in a neighborhood of  $x = 0$ . Thus,  $u_B(x, t)$ , as well as  $u(x, t)$ , approaches zero as  $t$  tends to infinity. This indicates that the null solution of  $(B_1)$  is also stable from above.

## 5. References

- [1] J. Anderson. *Qualitative studies of a convective porous medium equation with a nonlinear forcing at the boundary*. Thesis, Iowa State University, 1989.
- [2] W. A. Day. *Extensions of a property of the heat equation to linear thermoelasticity and other theories*. Q. Appl. Math. **40** (1982), 319–330.
- [3] W. A. Day. *A decreasing property of solutions of parabolic equations with applications to thermoelasticity*. Q. Appl. Math. **41** (1983), 468–475.
- [4] A. Friedman. *Monotonic decay of solutions of parabolic equations with non-local boundary conditions*. Q. Appl. Math. **44** (1986), 401–407.
- [5] A. Friedman. *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, NJ, 1965.
- [6] B. Kawohl. *Remarks on a paper by W. A. Day on a maximum principle under nonlocal boundary conditions*. Q. Appl. Math. **44** (1987), 751–752.
- [7] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva. *Linear and Quasi-linear Equations of Parabolic Type*. Translations of Mathematical Monographs, **23**.
- [8] H. A. Levine. *Stability and instability for solutions of Burgers' equation with a semilinear boundary condition*. SIAM J. Math. Anal. **19** (1988), 312–336.
- [9] H. A. Levine and G. M. Lieberman. *Quenching of solutions of parabolic equations with nonlinear boundary conditions in several dimensions*. J. Reine Ang. Math. **345** (1983), 23–38.
- [10] H. A. Levine and L. E. Payne. *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time*. J. Differential Equations **16** (1974), 319–334.
- [11] G. M. Lieberman. *Mixed boundary value problems for elliptic and parabolic differential equations of second order*. J. Math. Anal. Appl. **113** (1986), 422–440.

- [12] H. Matano. *Strongly order-preserving local semidynamical systems — theory and applications.* in Semigroups, theory and applications. Vol. 1. Longman Sci. Tech., Harlow, 1986, 178–185.
- [13] S. R. Park. *The phenomenon of quenching in the presence of convection.* Thesis, Iowa State University, 1989.

## GENERAL SUMMARY

In this thesis, we have investigated the asymptotic behavior of solutions of three nonlinear initial-boundary value problems of parabolic type.

In the first part, we proved that for a problem with a singular nonlinearity in the equation, if the solution quenches, then the quenching points are in a compact subset and the time derivative blows up.

In the second part, we considered a fast diffusion equation. We constructed the set of stationary solutions and determined when the solutions might or might not quench.

In the third part, we studied several nonlocal problems for Burgers' equation. We established the comparison principle and the existence of solutions and completed the characterizations of the steady states. We also gave the criteria for stability and instability and showed blow up results for some solutions.

---

## LITERATURE CITED

- [1] J. G. Berryman and C. J. Holland., *Stability of the separable solution for fast diffusion*. Arch. Rat. Mech. Anal. **74** (1980), 379–388.
- [2] P. G. Drazin and W. H. Reid. Hydrodynamic stability. Cambridge University Press, Cambridge, 1982.
- [3] F. John. Partial Differential Equations. Fourth edition. Springer-Verlag, New York, 1982.
- [4] H. Kawaiada. *On the solutions of initial- boundary value problems for  $u_t = u_{xx} + \frac{1}{1-u}$* . Publ. RIMS Kyoto Univ. **10** (1975), 729–736.
- [5] H. A. Levine. *Stability and instability for solutions of Burgers' equation with a semilinear boundary condition*. SIAM J. Math. Anal. **19** (1988), 312–336.
- [6] H. Lewy. *An example of a smooth linear partial differential equation without solution*. Ann. Math. **66** (1957), 155–158.
- [7] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*. Grundlehren der Mathematischen Wissenschaften **258**. Springer-Verlag, New York, 1983.
- [8] A. N. Tikhonov and V. Y. Arsenin. Solution of Ill-Posed Problems. Winston/Wiley, New York, 1977.

### ACKNOWLEDGEMENTS

I want to thank my parents for their constant encouragement and firm support. I thank Dr. James C. Hill, Dr. Elgin H. Johnston, Dr. Paul E. Sacks, and Dr. Stephen Willson for serving on my committee and for guiding me through my graduate education. I am also grateful to all the Mathematics office secretaries—Madonna, Jan, Ruth, Joanne, and Connie—for their kind help. I should like to express my heartfelt thanks to Dr. Howard A. Levine, my major professor, for his profound concern, patient guidance, and warm friendship. I rejoice to have such a good teacher and helpful friend. I am especially grateful to my wife, Jianhong Wang, for her deep love and complete understanding.